A Characterization of Affine Minimal and Affine Flat Surfaces

by

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A Characterization of Affine Minimal and Affine Flat Surfaces

Thesis directed by Professor Jeanne Clelland

The primary defining characteristic of Euclidean geometry in $\mathbb{R}^3$ is the presence of a flat metric, $\langle , \rangle$ which is defined on all tangent vectors to all points in $\mathbb{R}^3$ and invariant under the action of the Euclidean group. When studying submanifolds of the Euclidean space $\mathbb{E}^3$ (i.e., $\mathbb{R}^3$ together with a Euclidean metric), all metric properties (e.g., arc lengths and surface areas) are derived from this underlying metric. By contrast, in equiaffine geometry (which, for convenience, we will refer to simply as "affine geometry"), it is not possible to define a metric on tangent vectors which is preserved by the action of the equiaffine group. There is an invariant volume form, but no invariant notion of distance which can be restricted to submanifolds of $\mathbb{A}^3$ (i.e., $\mathbb{R}^3$ together with an equiaffine structure, which we will define shortly) in any obvious way.

Nevertheless, it is possible to define a notion of affine metric for generic surfaces in such a way that this notion is preserved by the action of the equiaffine group. Because there is no inner product on tangent vectors, this affine notion of metric on submanifolds depends on higher-order derivatives, as opposed to the analogous Euclidean notion, which depends only on the first derivatives of a surface.

Once we construct a measure of distance on an affine surface, we implicitly construct a notion of curvature on a surface–i.e., how that surface bends and changes. From this, the notions of affine minimal surfaces and affine flat surfaces emerge [1]. An affinely flat surface is a surface with zero affine curvature. An affinely minimal surface is the affine analogue to a Euclidean minimal surface—a surface that has locally extremal surface area. In this work, we characterize hyperbolic[1] affine surfaces that are both affine minimal and affine flat.

[1] Defined in chapter 2.

To Alexandra Fresch

Without your emotional support throughout my college career and your presence as a role model, I might never have pursued an honors thesis.

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Finally, I would like to thank my parents and my grandparents, whose continued love, encouragement, support, and belief in my abilities have made it possible for me to go from a very mediocre student to an honors student, and who have always encouraged me to ask as many questions about the world as possible.

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Introduction

The primary goals of this thesis are two-fold:

(1) To characterize affine flat, affine minimal hyperbolic affine surfaces using the method of moving frames.

(2) To develop enough background that the statement of (1) makes sense.

The first three chapters are devoted to (2). In chapter 1, we introduce important ideas in broad strokes. In chapter 2, we develop the bulk of the formalism required to do moving frames in equiaffine space. In chapter 3, we develop some general facts about surfaces in equiaffine space. In chapter 4, we use what we learned to actually characterize our surfaces. Finally, in chapter 5, we describe what this characterization might be useful for.

In this chapter, we introduce essential notions in differential geometry, we roughly describe equiaffine space, and we introduce the idea of *moving frames.* Although many notions in this section are very basic, they are essential parts of the philosophy of moving frames. For this reason, we briefly address many simple ideas.

1.1 Differential Geometry

The study of differential geometry is, as the name implies, the study of change. Geometers use calculus to study the properties of geometric objects. Differential geometry very much owes its heritage to the ancient Greeks. Greek geometers studied only shapes they could draw—two-dimensional objects like triangles, circles, and squares. Differential geometers, however, study geometry in any number of dimensions: spheres, polygons, hypercubes, 4-dimensional spacetimes, and even more exotic objects.

These objects are quite difficult to study directly. To more easily understand them, geometers borrow an essential technique from differential calculus: the linear approximation.

1.1.1 The Linear Approximation

In calculus, the tangent line of a curve $y = f(x)$ (at a given point $\mathbf{p}$) is defined as the a line based at $\mathbf{p}$ with slope equal to the derivative of our curve, $\frac{\Delta y}{\Delta x} = f'(x)$. We think of the tangent line as representing the graph of $f(x)$ at points near $\mathbf{p}$ very roughly. Because the real numbers are complete, we can "zoom in" to look as closely at the graph of $f(x)$ as we like. If we zoom in close enough, the graph looks like a line: the tangent line. This is what we call a *linear approximation.*

We call any object that can be linearly approximated a *manifold.* For completeness, we present the definition of manifold here. However, informally manifolds should be treated as geometric objects for which a linear approximation makes sense.

Definition 1.1.1 Manifold A topological space M is a *manifold* if it satisfies the following properties:

- The collection of open sets on M has a countable basis. We can write down a countable collection of open sets in M, call it $\mathcal{B}$ such that every open set in M can be written as a union of elements in $\mathcal{B}$.
- Given any two points $x, y \in M$, there exist neighborhoods of x and y, call them $N(x)$ and $N(y)$, such that $x \in N(x)$, $y \in N(y)$, and $N(x) \cap N(y) = \emptyset$. Roughly, this condition states that every point is separate from every other point.
- M is locally homeomorphic to $\mathbb{R}^n$, where n is the dimension of M.

The really important piece of information is the last requirement. This tells us that M has a tangent space, and a manifold is really defined by the property that it has a tangent space.

1.1.2 The Tangent Space

We can generalize the tangent line to a variety of objects, including surfaces, hypersurfaces, and more. For simplicity, let's consider a surface living in three-dimensional Euclidean space. We want to study the surface at a point. If we zoom in and look at the surface closely enough, it looks like a plane, a copy of $\mathbb{R}^2$. This is the two-dimensional analogue of the tangent line. The plane is called the "tangent plane," or the "tangent space." It is the linear approximation of a surface at a given point.

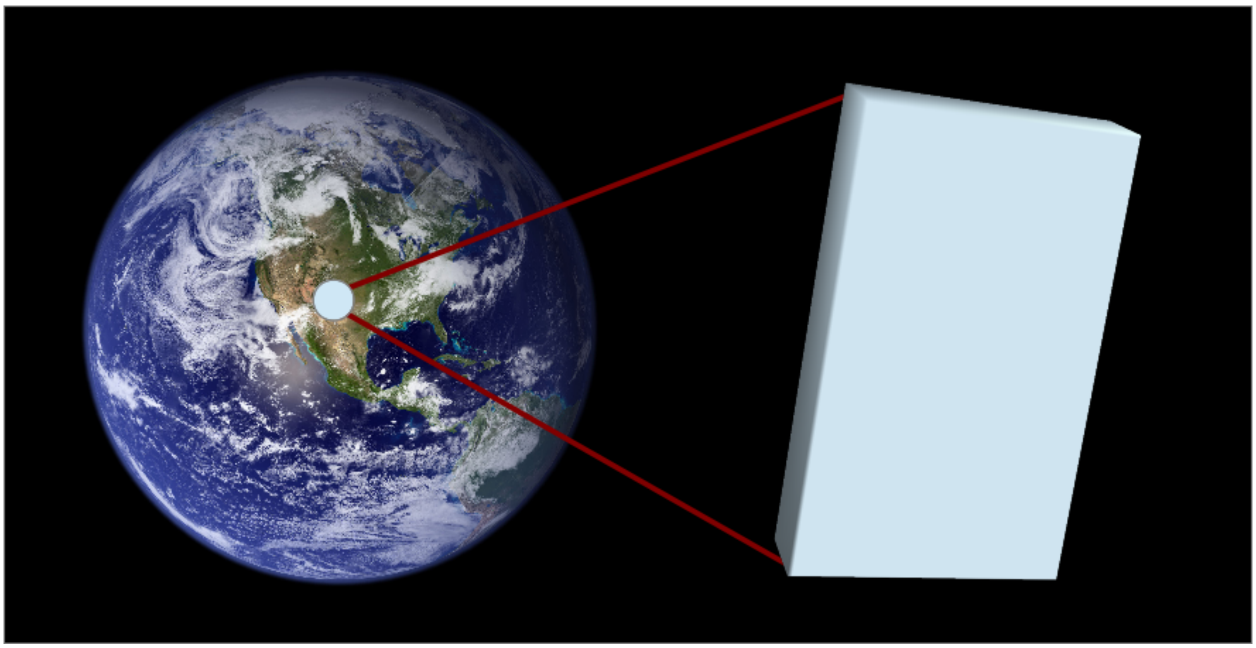

Figure 1.1: While living on it, we see the Earth as flat, even though it is spherical. This is because we see the tangent plane of the earth. We've "zoomed in" so close that all we see is the linear approximation, and we miss the global structure. Image of the Earth from [4].

In general, the tangent space based at a point $\mathbf{p}$ is defined as the space containing the tangent vectors of all curves contained in the manifold that pass through $\mathbf{p}$.

Definition 1.1.2 Let M be a differential manifold and let $\mathbf{p}$ be a point on M. The **tangent space** of M at $\mathbf{p}$, denoted $T_{\mathbf{p}}M$, is the set of all tangent vectors at $\mathbf{p}$ to all curves contained in M that pass through $\mathbf{p}$.

As we shall see, the method of linear approximation is essential to differential geometry.

1.2 Measuring Distance

One way we use the method of linear approximation is in the measurement of distance. Most of us take distance for granted. We take it as intuitively obvious that a circle has a radius r and that it has a circumference of $c = 2\pi r$. However, measuring distance is not always easy, or even possible. In regular Euclidean space (the world we live in), we meausure distance by calculating the lengths of vectors in the tangent space.

The following example describes how we measure the length of a curve—from which we can get the distance between two points—in three-dimensional space.

Example 1.2.1 Arc Length

Let

$$\alpha : I \to \mathbb{R}^3, \ I \subseteq \mathbb{R}$$

be a smooth curve in three-dimensional Euclidean space parametrized as a (row) vector of three functions:

$$\alpha(t) = [\alpha^1(t), \alpha^2(t), \alpha^3(t)],$$

where $\alpha^1, \alpha^2, \alpha^3$ are all infinitely differentiable functions from I to the real line. For a given $t \in I$, the tangent vector of α is

$$\alpha'(t) = [(\alpha^1)'(t), (\alpha^2)'(t), (\alpha^3)'(t)].$$

At any given t, the length of α's tangent vector, its *speed*, is

$$\mathtt{speed}_\alpha(t) = \|\alpha'(t)\| = \sqrt{\alpha'(t) \cdot \alpha'(t)},$$

where $\cdot$ is the usual vector dot product.

We can think of the line $L(t) = \alpha(t_0) + t\alpha'(t_0)$ as a linear approximation for α at time t_0. If $t_0 \in I$, then $L(t)$ is a very good approximation for α near t. In fact, it describes α perfectly at t_0. Thus, by looking at $L(t)$ at each point, we learn everything there is to know about α, and we learn it much more easily than if we had tried a non-calculus approach. For instance, if we want to know the length of α along the interval

$[t_1, t_2] \in I$, we can calculate the arc length:

$$S = \int_{t_1}^{t_2} \texttt{speed}_\alpha(t)dt = \int_{t_1}^{t_2} \sqrt{\alpha'(t) \cdot \alpha'(t)}dt.$$

And, if we want to find the distance between two points, **A** and **B**, we find the shortest possible path between them and calculate its arc length. The notion of a tangent space is critical for this formulation of distance.

1.3 Affine Space, When Distance Disappears

Most of this work will deal with a world where distance in the classical sense has no meaning. In Equiaffine Space (for convenience we will usually refer to it as *affine space*), we consider two objects equivalent if one can be transformed into the other by a linear volume-preserving transformation. For example, a sphere is considered equivalent to an ellipsoid of the same volume. It is not, however, considered equivalent to a cube of the same volume.

We will make this precise in chapter 2. However, in the meantime, let's see how this equivalence invalidates any traditional notion of distance.

Example 1.3.1 Consider the unit sphere S^2 embedded in affine space and place two points on the poles. Call them **A** and **B**. Before affine transformation, the sphere is of radius 1, so **A** and **B** have a distance of 2 between them. However, we can perform a volume-preserving transformation that stretches the sphere along the z-axis. The result is that the points are separated by a greater distance. Although the sphere LOOKS thinner to Euclidean eyes, it is actually the same sphere in affine space. We can use affine transformations to separate **A** and **B** by any distance we like.

As we shall see in chapter 2, there is a way to define the arclength of a curve in equiaffine space, but it is a very alien notion. Furthermore, there is no absolute sense of distance like there is in Euclidean space. Without a curve or a surface from which we can derive arclength, distance has no meaning. In the meantime, there's another important notion we need to introduce, that of the moving frame.

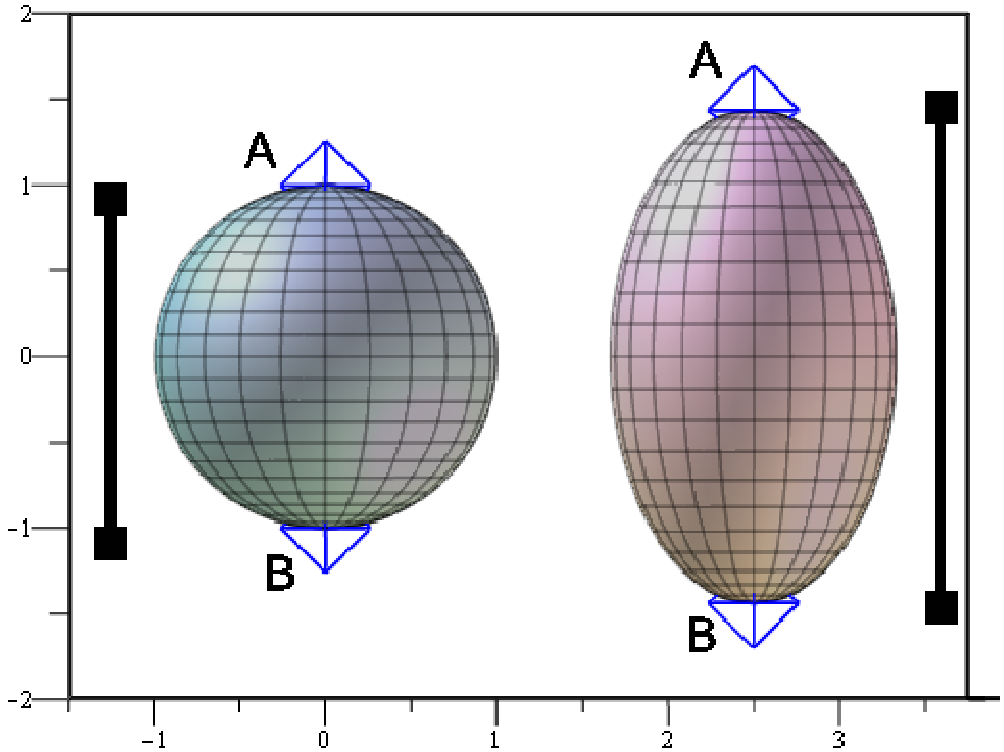

Figure 1.2: Using an affine transformation, we can make the distance between the poles of a sphere as great or as small as we like.

1.4 Moving Frames

In differential geometry, we attempt to build up an accurate picture of some geometric object entirely using linear approximations. We do this by choosing a *basis* for the tangent space of that object very carefully. Outside of moving frames, we usually use a *coordinate basis* like the Cartesian coordinates or spherical coordinates to specify a point's location in space. Each coordinate basis has its advantages. On the one hand, the Cartesian basis for Euclidean space has the very nice property of being orthonormal everywhere, i.e., for any two basis vectors $\underline{e}_i, \underline{e}_j$,

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}.$$

On the other hand, spherical coordinates make spherical symmetry very obvious and easy to deal with. As a cost, however, spherical coordinates don't make any sense at the origin.

In general coordinates in Euclidean space, we usually can't have a coordinate basis that's nicely orthonormal everywhere but also respects the symmetry of whatever geometric object we're studying. However, by taking a different approach—the method of moving frames—we can have our cake and eat it too. We choose a *non-coordinate basis* or *moving frame*, an orthonormal basis which varies continuously from point to point. Unlike with a coordinate basis, the basis vectors of a moving frame at one point do not have to be the same as the basis vectors at another. By choosing our basis vectors to reflect the properties of a geometric object of interest, we can fully describe that object... and learn a lot about it in the process. This technique is known as Cartan's *method of moving frames*, named after the discoverer.

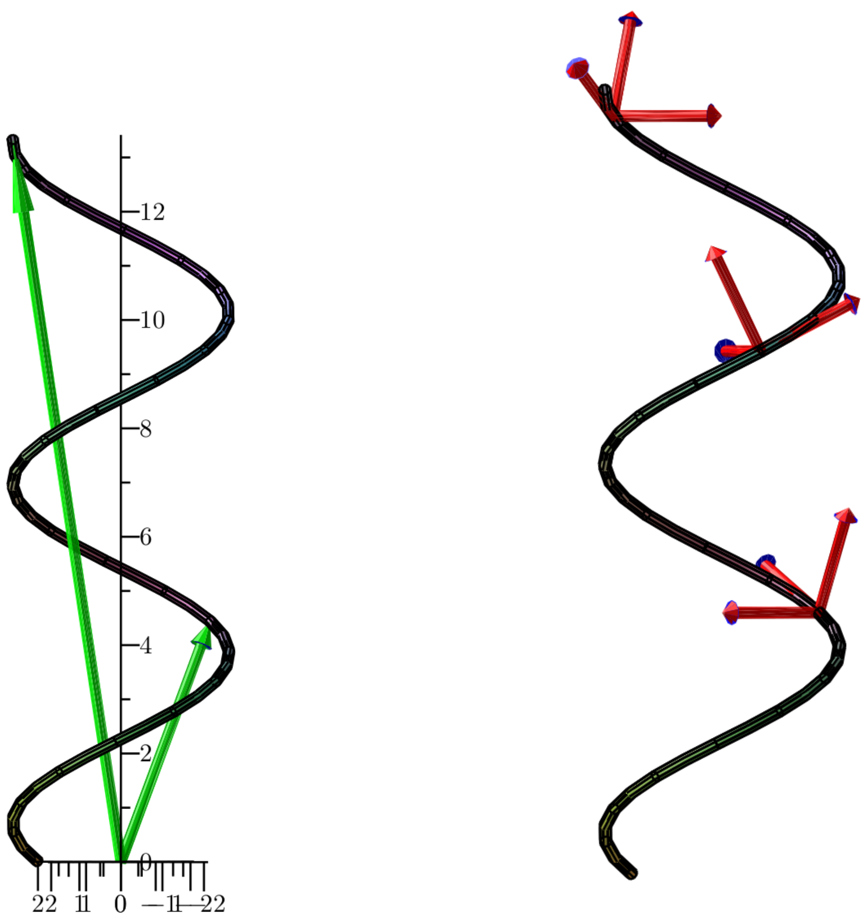

Figure 1.3: Rather than defining each point in relation to some central origin and globally-defined basis (left), a non-coordinate basis (right) defines a new set of basis vectors at each point.

Choosing a basis—coordinate or non-coordinate—to describe a curve or surface amounts to choosing a set of tangent vectors in the tangent space of that curve or surface. If the object is embedded in some larger space, such as $\mathbb{R}^3$, then we look at the tangent space of the underlying space because a lot of geometric information is contained there.

Although the notion of orthonormality only makes sense in Euclidean space,[1] moving frames are very general. If a notion of orthonormality doesn't exist, then we choose a moving frame based on other geometrically nice properties.

To get a better idea of how moving frames work, let's look at the simplest interesting case: a curve embedded in $\mathbb{E}^3$. The following treatment comes from [3].

1.4.1 The Frenet Frame

Let

$$\alpha : I \to \mathbb{R}^3, I \subseteq \mathbb{R}$$

be a smooth curve in three-dimensional Euclidean space parametrized as in example 1.2.1. For convenience and simplicity, we add the additional condition that α is *unit speed*. In other words, we impose that

$$\|\alpha'(t)\| \equiv 1. \tag{1.4.1}$$

Every curve has a unit speed parametrization (even if, as in the case of an ellipse, we can't explicitly write it down) so these results are quite general.[2]

We must go about choosing a set of orthonormal vectors to describe our curve at each point. This amounts to choosing three orthogonal unit vectors at each $t \in I$. A good first choice might be the curve's tangent vector, $\alpha'(t)$. Because α is unit speed, α' is already a unit vector. Thus, we choose our first "frame vector" to be

$$\mathbf{T} := \alpha'(t), \tag{1.4.2}$$

also called the "unit tangent vector."

A winning strategy in differential geometry is often to differentiate things: it's easy and it often leads to new insights. For this reason, throughout this work, we will keep in mind the following mantra:[3]

[1] Or spaces with similar structure

[2] Curves that can be parametrized in this way are called regular curves. A curve α is *irregular* if at any point $\|\alpha'(t)\| = 0$. If a curve is not regular, we can usually break it up into regular pieces.

[3] I learned this mantra from Professor Jeanne Clelland, and I have copied it word for word.

When in doubt, differentiate!

Let's see what insight we can gain from differentiating $\mathbf{T}$. Since it is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1.$$

Let's see what happens when we differentiate this equation.

$$\begin{aligned} 1 &= \mathbf{T}(t) \cdot \mathbf{T}(t) \\ \Rightarrow \frac{d}{dt}(1) &= \frac{d}{dt}\left(\mathbf{T}(t) \cdot \mathbf{T}(t)\right) \\ \Rightarrow 0 &= \mathbf{T}'(t) \cdot \mathbf{T}(t) + \mathbf{T}(t) \cdot \mathbf{T}'(t) = 2\mathbf{T}(t) \cdot \mathbf{T}'(t). \\ \Rightarrow \mathbf{T}'(t) \cdot \mathbf{T}(t) &= 0 \end{aligned}$$

So the unit tangent and its derivative are orthogonal! The second derivative of the curve is a natural choice for our second basis vector. However, it might not be normalized, so we need to fix that before we can use it as a basis vector. We don't want to lose information, so we'll keep track of that normalization factor. Define

$$\kappa(t) := \sqrt{\mathbf{T}'(t) \cdot \mathbf{T}'(t)}. \tag{1.4.3}$$

If $\kappa(t) = 0$ at a given point, $\mathbf{T}$ isn't changing direction and our curve resembles a straight line at time t. If $\kappa(t) \neq 0$, let

$$\mathbf{N}(t) := \frac{1}{\kappa(t)}\mathbf{T}'(t). \tag{1.4.4}$$

We call $\kappa(t)$ the *curvature* and $\mathbf{N}(t)$ the *normal vector* of the curve α. Now that we have two orthogonal unit vectors, we can easily find a third: just take the cross product of the first two:

$$\mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t). \tag{1.4.5}$$

We call this the *binormal vector.*

Now that we have a coordinate-free basis, let's see what we can learn from it. Because $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ form an orthonormal basis for $\mathbb{E}^3$ along α, we can write any vector as a linear combination of them. In particular, we can use them as a basis for their derivatives. Remember, when in doubt, differentiate:[4]

$$\mathbf{B}' = a\mathbf{T} + b\mathbf{N} + c\mathbf{B},$$

[4] For convenience, we suppress the time-dependence of the frame vectors in our notation. However, it's important to realize that this time-dependence is there.

where $a, b, c \in \mathbb{R}$ are scalar functions of t. If we can calculate a, b and c, we can find out what $\mathbf{B}'$ is. To do this, we project $\mathbf{B}'$ onto our basis vectors.

$$\mathbf{T} \cdot \mathbf{B}' = a\mathbf{T} \cdot \mathbf{T} + b\mathbf{T} \cdot \mathbf{N} + c\mathbf{T} \cdot \mathbf{B} = a$$

Similar calculations reveal that $\mathbf{N} \cdot \mathbf{B}' = b$ and $\mathbf{B} \cdot \mathbf{B}' = c$. Thus,

$$\mathbf{B}' = (\mathbf{T} \cdot \mathbf{B}')\mathbf{T} + (\mathbf{N} \cdot \mathbf{B}')\mathbf{N} + (\mathbf{B} \cdot \mathbf{B}')\mathbf{B}.$$

To calculate these coefficients, we take advantage of the orthogonality of our basis vectors. For example, we know that

$$\mathbf{T} \cdot \mathbf{B} = 0.$$

When in doubt, differentiate.

$$\begin{aligned}
0 &= \mathbf{T} \cdot \mathbf{B} \\
\Rightarrow \frac{d}{dt} 0 &= \frac{d}{dt} [\mathbf{T} \cdot \mathbf{B}] \\
\Rightarrow 0 &= \mathbf{T}' \cdot \mathbf{B} + \mathbf{T} \cdot \mathbf{B}' \\
\Rightarrow \mathbf{T} \cdot \mathbf{B}' &= -\mathbf{T}' \cdot \mathbf{B} \\
&= -\kappa \mathbf{N} \cdot \mathbf{B} \text{ because } \mathbf{T}' = \kappa \mathbf{N} \\
&= 0
\end{aligned}$$

because $\mathbf{N}$ and $\mathbf{B}$ are orthonormal. We can play a similar trick to find $\mathbf{B} \cdot \mathbf{B}'$:

$$\begin{aligned}
1 &= \mathbf{B} \cdot \mathbf{B} \text{ by orthonormality} \\
\Rightarrow \frac{d}{dt} 1 &= \frac{d}{dt} (\mathbf{B} \cdot \mathbf{B}) \\
\Rightarrow 0 &= \mathbf{B}' \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B}' = 2\mathbf{B} \cdot \mathbf{B}' \\
\Rightarrow \mathbf{B} \cdot \mathbf{B}' &= 0.
\end{aligned}$$

We can't find $(\mathbf{N} \cdot \mathbf{B}')$ using this trick, and there's no immediately obvious way to constrain it in general. Since we can't constrain this term, we give it a name instead. We define

$$\tau(t) = -\mathbf{N}(t) \cdot \mathbf{B}'(t) \tag{1.4.6}$$

to be the *torsion* of $\alpha(t)$. Thus,

$$\mathbf{B}' = -\tau\mathbf{N}.$$

We can play a similar game to find $\mathbf{N}'$. By projecting it onto the $\mathbf{T}, \mathbf{N}, \mathbf{B}$ basis, we find that

$$\mathbf{N}' = (\mathbf{T} \cdot \mathbf{N}')\mathbf{T} + (\mathbf{N} \cdot \mathbf{N}')\mathbf{N} + (\mathbf{B} \cdot \mathbf{B}')B.$$

Then, since $\mathbf{T} \cdot \mathbf{N} = 0$,

$$\mathbf{T}' \cdot \mathbf{N} = -\mathbf{T} \cdot \mathbf{N}'.$$

Since $\mathbf{T}' = \kappa\mathbf{N}$,

$$\mathbf{T} \cdot \mathbf{N}' = -\kappa\mathbf{N} \cdot \mathbf{N} = -\kappa.$$

Since $\mathbf{N} \cdot \mathbf{N} = 1$, $\mathbf{N} \cdot \mathbf{N}' = 0$. Since $\mathbf{B} \cdot \mathbf{N} = 0$,

$$\begin{aligned} 0 &= \mathbf{B}' \cdot \mathbf{N} + \mathbf{B} \cdot \mathbf{N}' \\ \Rightarrow \mathbf{B} \cdot \mathbf{N}' &= -\mathbf{B}' \cdot \mathbf{N} = -\mathbf{N} \cdot \mathbf{B}' \\ &= \tau. \end{aligned}$$

So, we find that

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

When we put it all together, we get a coupled system of ordinary differential equations for $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$:

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{1.4.7}$$

These are called the *Frenet equations.* Note that although this is a matrix equation, the coefficients in each vector are themselves vector valued. Each one is a basis vector field.

1.4.2 Applications of the Frenet Frame

Although we may not know the values of $\kappa(t)$ and $\tau(t)$, if we are given these values, we can solve the Frenet equations to find the equation for a curve $\alpha(t)$. Indeed, not only can we find such a curve, but the existence/uniqueness theorems for ordinary differential equations guarantee that our curve will be unique up to rigid motion.[5] Because $\kappa(t)$ and $\tau(t)$ remain the same when the curve is transformed by a rigid motion, they are called *invariants*.

When studying a geometric object, a typical strategy in the method of moving frames is to choose a non-coordinate basis so that a set of invariants for that object become apparent. We can then use those invariants to uniquely describe that object and to discover its properties. Because a moving frame ignores general coordinates, any quantities we find using this method will always be invariant under appropriate transformations.

The following three simple examples demonstrate how we can use invariants to help us learn about a curve.

Example 1.4.1 $\kappa(t) = \tau(t) = 0 \ \forall \ t \in I$.

If we let $\kappa = \tau = 0$, the Frenet equations become just one relevant equation:

$$\mathbf{T}'(t) = 0.$$

If we rewrite this in terms of α, we have

$$\alpha''(t) = 0.$$

We can integrate this twice to find that

$$\alpha(t) = \mathbf{a}t + \mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. So the unique curve with $\kappa = \tau = 0$ is a line.

Example 1.4.2 $\kappa(t) = c$ and $\tau(t) = 0 \ \forall \ t \in I$.

If we let $\kappa(t)$ be some constant positive value, call it c and $\tau = 0$, the Frenet equations become

$$\begin{aligned} \mathbf{T}'(t) &= c\mathbf{N}(t) \\ \mathbf{N}'(t) &= -c\mathbf{T}(t) \\ \mathbf{B}'(t) &= \vec{0}. \end{aligned}$$

[5] Rigid motion is, of course, translations and rotations.

Since $\mathbf{N}$ and $\mathbf{T}$ are independent of $\mathbf{B}$, we can solve their coupled equations to find $\alpha(t)$. This is a simple enough system that we can guess the solution. In Cartesian coordinates and up to rigid motion, we have:

$$\mathbf{T}(t) = \begin{bmatrix} -\sin(ct) \\ \cos(ct) \\ 0 \end{bmatrix} \quad \texttt{and} \quad \mathbf{N}(t) = \begin{bmatrix} -\cos(t) \\ -\sin(ct) \\ 0 \end{bmatrix}.$$

If we set $\alpha'(t) = \mathbf{T}(t)$ and integrate, we find that α is

$$\alpha(t) = \begin{bmatrix} \frac{1}{c}\cos(ct) \\ \frac{1}{c}\sin(ct) \\ 0 \end{bmatrix}.$$

In other words $\alpha(t)$ is a circle of radius $1/c$. We begin to see why $\kappa(t)$ is called curvature. At any given time t_0, $\kappa(t_0)$, $\mathbf{N}(t_0)$ and $\alpha(t_0)$ describe a circle, and $1/\kappa(t_0)$ gives us that circle's radius.

Example 1.4.3 $\kappa(t) = c_1$ and $\tau(t) = c_2 \ \forall \ t \in I$.

If we let $\kappa(t)$ and $\tau(t)$ be two positive constants, call them c_1 and c_2, we must solve all three Frenet equations. However, because our defining functions are constants, the equations are still relatively simple.

$$\begin{aligned} \mathbf{T}'(t) &= c_1\mathbf{N}(t) \\ \mathbf{N}'(t) &= -c_1\mathbf{T}(t) + c_2\mathbf{B}(t) \\ \mathbf{B}'(t) &= -c_2\mathbf{N}(t) \end{aligned}$$

In this case, the Frenet equations yield

$$\mathbf{T}(t) = \begin{bmatrix} \frac{c_1}{\sqrt{c_1^2+c_2^2}}\sin(\sqrt{c_1^2+c_2^2}t) \\ -\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\sqrt{c_1^2+c_2^2}tt) \\ \frac{c_2}{c_1} \end{bmatrix}, \ \mathbf{N}(t) = \begin{bmatrix} \cos(\sqrt{c_1^2+c_2^2}t) \\ \sin(\sqrt{c_2^2+c_2^2}t) \\ 0 \end{bmatrix}, \ \texttt{and} \ \mathbf{B}(t) = \begin{bmatrix} -\frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\sqrt{c_1^2+c_2^2}t) \\ \frac{c_2}{\sqrt{c_1^2+c_2^2}}\cos(\sqrt{c_1^2+c_2^2}t) \\ 1 \end{bmatrix}.$$

If we integrate $\mathbf{T}$, we get a helix:

$$\alpha(t) = \begin{bmatrix} -\frac{c_1}{c_1^2+c_2^2}\cos(\sqrt{c_1^2+c_2^2}t) \\ -\frac{c_1}{\sqrt{c_1^2+c_2^2}}\sin(\sqrt{c_1^2+c_2^2}t) \\ \frac{c_2}{c_1}t \end{bmatrix}.$$

Now we begin to get an idea of what κ and τ mean. While κ controls how bent in-plane a curve is, τ controls how three-dimensional a curve is. When $\tau = 0$, a curve is confined to a plane. When τ is nonzero, the curve moves out of the plane. The Frenet frame allows us to easily characterize curves in terms of these two invariants.

The ease with which we were able to gain all of this information is mostly due to our ability to write the derivatives of each frame vector as a linear combination of the other vectors. This allows us to compare the vectors and build our integrable system. This is the core technique of the method of moving frames. However, as we shall see, it is generalizable to a wide variety of geometric objects.

Chapter 2

Background

Before we can proceed, there are a number of important concepts required to understand the method of moving frames in equiaffine space. We briefly develop these ideas here.

2.1 Vectors and Covectors

As we discussed in chapter 1, a central idea in differential geometry is the idea of linear approximation. For this reason, the *tangent vector* is especially important.

Definition 2.1.1 Geometric Vector Given a manifold M, a *geometric vector* at a given point $\mathbf{p}$ is an element of the tangent space of M based at the point $\mathbf{p}$.

Remark 2.1.2 Although every vector has both direction and magnitude, these quantities may not be invariant or meaningful. We will make this precise in section 2.7.2.

The tangent space of a manifold based at a point $\mathbf{p}$ forms a vector space in the linear algebra sense.[1] Thus we can find some set of linearly independent tangent vectors that form a *basis* for the tangent space at that point. We call the tangent space of M based at $\mathbf{p}$ $T_{\mathbf{p}}M$ [2].

Remark 2.1.3 In general, the way we approximate a manifold at one point will be different from the way we approximate a manifold at a different point. For this reason, given two different points on a manifold, $\mathbf{p}$

[1] In the case of a curve, the tangent space is just the line spanned by the tangent vector. In the case of a surface, it is a plane.

and $\mathbf{q}$, the tangent spaces $T_{\mathbf{p}}M$ and $T_{\mathbf{q}}M$ really are different vector spaces.[2]

In mathematics, whenever we construct an object, it is helpful to know how to measure that object. For this reason, we develop the notion of a *covector*. We use covectors to measure the size of vectors—we can loosely think of them as meter sticks for vectors [2].

Definition 2.1.4 Covector Let V be an n-dimensional vector space. A *covector* is a linear map from V to the real line.

In other words, we can think of a covector as a function that takes tangent vectors as input and returns a real number.

Definition 2.1.5 The set of all covectors based at a point $\mathbf{p}$ on a manifold M is called the *cotangent space* at $\mathbf{p}$. We denote it $T^*_{\mathbf{p}}M$.

Remark 2.1.6 Covectors are *dual* to vectors. We could easily treat covectors as the base object, and vectors as maps from the cotangent space to the real line. In other words, we can treat vectors as meter sticks for covectors. This means that covectors are just as good for linear approximation as vectors are [2].

Definition 2.1.7 The set of all covectors for a given vector space V is called the *dual space* of V and it is denoted V^*.

2.2 Tensors

We can use vectors and covectors to develop higher-order geometric objects, which are made up of both vectors and covectors. To do this, we introduce the notion of the *tensor product.* To avoid confusion, we will be very informal here. (Definition from [2].)

Definition 2.2.1 Tensor Product Let V and W be vector spaces and let $\alpha \in V^*$ and $\beta \in W^*$ be covectors. The *tensor product* of α and β is the function from $V \times W$ to the real line

$$(\alpha \otimes \beta) : V \times W \to \mathbb{R}$$

[2] That is not to say that the tangent spaces aren't related. The set of all tangent spaces of a manifold M form a *tangent bundle* and (if we have a notion of the straightness of lines) we can translate between using something called *parallel transport.*

defined by

$$(\alpha \otimes \beta)(\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{v})\beta(\mathbf{w}),$$

where $\mathbf{v} \in V$ and $\mathbf{w} \in W$.

To define the tensor product for vectors, we simply take advantage of the fact that vectors are dual to covectors. The tensor product of two vectors is just a map from the direct product of the covector spaces to the real line.

More informally, if $\mathbf{v}_1, ...\mathbf{v}_n$ are vectors and $\alpha_1, ...\alpha_m$ are covectors, then

$$\mathbf{v}_1 \otimes ... \otimes \mathbf{v}_n \otimes \alpha_1 \otimes ... \otimes \alpha_m(\beta_1, ...\beta_n, \mathbf{w}_1, ..., \mathbf{w}_m,) = \mathbf{v}_1(\beta_1)...\mathbf{v}_n(\beta_n)\alpha(\mathbf{w}_1)...\alpha(\mathbf{w}_m), \quad (2.2.1)$$

where β_i are covectors and $\mathbf{w}_i$ are vectors. In other words, the tensor product combines vectors and covectors into higher-order objects where the vectors and covectors within the tensor product do not interact with each other.

Definition 2.2.2 Tensor A *tensor* is a linear combination of tensor products of vectors and covectors.

Definition 2.2.3 Tensor rank The *rank* of a tensor is an ordered pair (n, p), where n is the number of vectors in the tensor product and p is the number of covectors. A *rank zero tensor*, also called a *scalar*, is just a number.

A useful notion is the antisymmetric tensor product, or the *wedge product*.

Definition 2.2.4 Wedge Product Let α and β be rank 1 tensors. The *wedge product* of α and β is

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha. \quad (2.2.2)$$

This antisymmetrizes the inputs to the tensor [2].

Similarly, we can define the *symmetric tensor product*.

Definition 2.2.5 Let α and β be rank 1 tensors. The *symmetric tensor product* of α and β is

$$\alpha \circ \beta = \frac{1}{2} \left[\alpha \otimes \beta + \beta \otimes \alpha \right]. \quad (2.2.3)$$

Both definitions can be generalized to higher-rank tensors in the obvious way.

2.3 Differential Forms

2.3.1 1-Forms

In multivariable calculus, a typical chain rule calculation looks like this [2]:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

df, dx, dy, and dz are all *1-forms.* Roughly speaking, a 1-form is just a covector. However, 1-forms can be generated using the *exterior derivative*, which we will define in a moment, and form the base for constructing a special type of higher-order tensor, called a *p-form.* This makes them powerful tools [2].

Definition 2.3.1 1-Form A *smooth 1-form* ϕ on $\mathbb{R}^n$ is a map from the set of all tangent vectors in $\mathbb{R}^n$ to the real line

$$\phi : T\mathbb{R}^n \to \mathbb{R}$$

with the properties that

- For each $\mathbf{x} \in \mathbb{R}$, ϕ is a linear map from all vectors based at $\mathbf{x}$ to the real line.
- For any smooth[3] vector field, $\mathbf{v} = \mathbf{v}(\mathbf{x})$ on $\mathbb{R}^n$, the function $\phi(\mathbf{v}) : \mathbb{R}^n \to \mathbb{R}$ is smooth.

Let M be a manifold imbued with local coordinates x^i, then the 1-forms dx^i form a basis for the cotangent space of M at each point. The duals of 1-form basis elements form a geometrically natural basis for the tangent space.

Remark 2.3.2 The vectors dual to the covectors dx^i are $\frac{\partial}{\partial x^i}$ and these form a basis for the tangent space of M at a given point $\mathbf{p}$. For simplicity, we often write

$$\frac{\partial}{\partial x^i} = \partial_i. \tag{2.3.1}$$

Convention 2.3.3 To keep track of which geometric objects are vectors and which objects are 1-forms, we always index tangent vectors with the index down and cotangent 1-forms with the index up. Thus, dx^i is a 1-form, but ∂_i is a tangent vector.

[3] i.e., infinitely differentiable

2.3.2 Einstein Summation Notation

This property motivates the definition of *Einstein summation notation.* Since dx^i form a basis for the cotangent space and ∂x_i form a basis for the tangent space, we can write an n-dimensional covector as the sum of components times the basis elements

$$\alpha = \sum_{i=1}^{n} a_i dx^i. \tag{2.3.2}$$

Similarly, we can write a vector as the sum of the components times the basis elements.

$$\mathbf{v} = \sum_{i=1}^{n} v^i \partial x_i. \tag{2.3.3}$$

Convention 2.3.4 Einstein Summation Notation For convenience, we often surpress the summation symbol and allow summation to be implied whenever an up index and a down index share a symbol.

$$a_i b^i = \sum_{i=1}^{n} a_i b^i. \tag{2.3.4}$$

2.3.3 p-Forms

Simply put, a *p-form* is a higher-order tensor constructed out of 1-forms. In essence, a p-form is a totally antisymmetric $(0, p)$ tensor. p-forms have their own algebra, where the wedge product substitutes for multiplication. Any p-form Φ on $\mathbb{R}^n$ can be written as

$$\Phi = \sum_{(i_1,\ldots,i_p)} f_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{2.3.5}$$

2.3.4 The Exterior Derivative

We can generate new p-forms using the *exterior derivative.* The exterior derivative generalizes and formalizes the derivative from multivariable calculus. We first define the exterior derivative on a real-valued function. We then generalize it to any p-form.

Definition 2.3.5 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then the *exterior derivative* of f is the 1-form df with the property that

$$df_{\mathbf{x}}(\mathbf{v}) = \mathbf{v}(f) \ \forall \ \mathbf{x} \in \mathbb{R}^n, \ \mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n, \tag{2.3.6}$$

where $\mathbf{v}(f)$ is the directional derivative of f in the direction $\mathbf{v}$ [2].

Remark 2.3.6 The coordinate 1-forms dx^i are generated by applying the exterior derivative d to each coordinate function x^i.

The exterior derivative behaves as we would expect [2].

Theorem 2.3.7 Let f and g be differentiable maps from $\mathbb{R}^n$ to $\mathbb{R}$ and let h be a differentiable map from $\mathbb{R}$ to $\mathbb{R}$. The exterior derivative d has the following properties.

$$df = \frac{\partial f}{\partial x^i} dx^i = \partial_i f dx^i. \tag{2.3.7}$$

$$d(fg) = gdf + fdg. \tag{2.3.8}$$

$$d(h(f)) = \frac{dh}{df} df. \tag{2.3.9}$$

To generate higher-order p-forms, we define the exterior derivative so that it takes p-forms to $(p+1)$-forms [2].

Definition 2.3.8 Let Φ be the p-form defined as

$$\Phi = \sum_{(i_1,\ldots,i_p)} f_{i_1,\ldots,i_p}(\mathbf{x}) dx^{i_1} \wedge \ldots \wedge dx^{i_p}.$$

The *exterior derivative* $d\Phi$ of Φ is the $(p+1)$-form

$$d\Phi = \sum_{(i_1,\ldots,i_p)} df_{i_1,\ldots,i_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{2.3.10}$$

Because of the antisymmetry in the indices, if Φ is a p-form and Ψ is a q-form, then the product rule takes the form [2]

$$d(\Phi \wedge \Psi) = d\Phi \wedge \Psi + (-1)^p \Phi \wedge d\Psi. \tag{2.3.11}$$

Because mixed partial derivatives commute, we have the following powerful result.

Theorem 2.3.9 Let Φ be any differential form. Then

$$d(d\Phi) = 0. \tag{2.3.12}$$

Remark 2.3.10 The exterior derivative is what motivates the notation for an integral

$$F(x) = \int f(x)dx.$$

The dx is a 1-form. This is why the following manipulations makes sense:

$$u = x^2 + 2 \Rightarrow du = 2xdx$$

and

$$\frac{dy}{dx} = 2z \Rightarrow dy = 2zdx.$$

2.4 Derivatives of Maps Between Manifolds

Armed with the concept of an exterior derivative, we'd like to be able to differentiate more exotic objects.

Definition 2.4.1 Let $F : M \to N$ be a differentiable function between smooth manifolds M and N. Then the derivative of F at a point $\mathbf{p}$ is the unique linear map between tangent spaces

$$dF_{\mathbf{p}} : T_{\mathbf{p}}M \to T_{F(\mathbf{p})}N \tag{2.4.1}$$

such that for any curve $\alpha : \mathbb{R} \to M$ with $\alpha(0) = \mathbf{p}$ and $\alpha'(0) = \mathbf{v}$,

$$dF_{\mathbf{p}}(\mathbf{v}) = (F \circ \alpha)'(0). \tag{2.4.2}$$

This allows us to write the tangent space of a manifold as the image of the derivative of the parametrization of that manifold [2].

Remark 2.4.2 As we would expect, dF is a vector-valued 1-form. If $\{\mathbf{f}_1, ..., \mathbf{f}_n\}$ form a basis for the tangent space of N at each point $\mathbf{q} \in N$, then there exists a collection of scalar-valued 1-forms $\{\phi^1, ..., \phi^n\}$ on M such that

$$dF = \mathbf{f}_i \phi^i. \tag{2.4.3}$$

Intuitively, this is because dF is a function that eats vectors and returns vectors [2].

A simple example will help.

Example 2.4.3 Let Σ be a regular orientable surface embedded in three-dimensional Euclidean space. We parametrize Σ by a vector-valued function

$$\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$$

such that

$$\mathbf{X}(u,v) = \begin{bmatrix} f^1(u,v) \\ f^2(u,v) \\ f^3(u,v) \end{bmatrix},$$

where f^1, f^2, and f^3 are each scalar-valued functions of two variables.

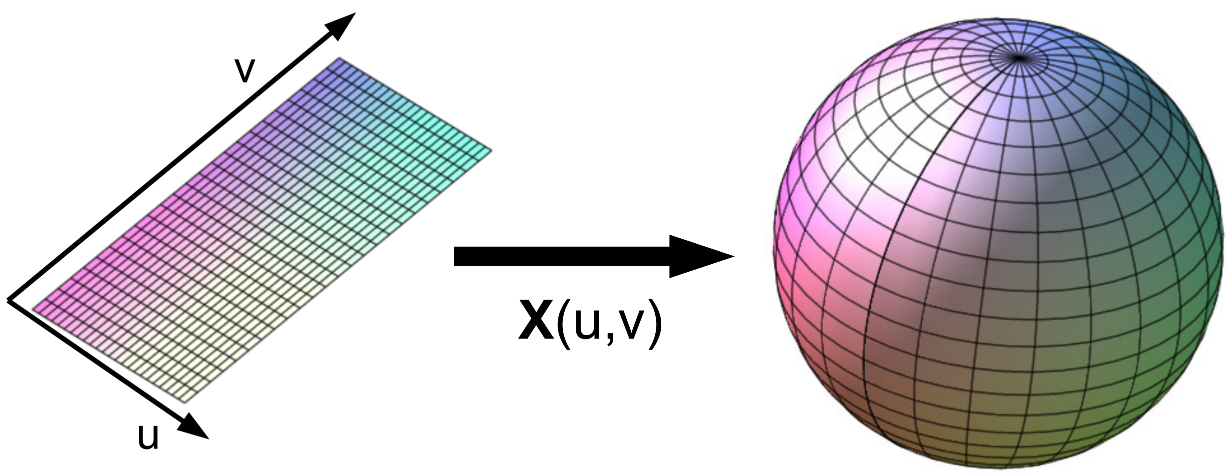

Figure 2.1: $\mathbf{X}$ maps $\mathbb{R}^2$ (left) to a surface embedded in $\mathbb{R}^3$ (right).

Then for each point $\mathbf{p} = (u_0, v_0) \in \mathbb{R}^2$, the exterior derivative of $\mathbf{X}$ is the linear map between tangent spaces $dX_{\mathbf{p}} : T_{\mathbf{p}}\mathbb{R}^2 \to T_{\mathbf{X}(\mathbf{p})}\mathbb{R}^3$. Assuming f^1, f^2 and f^3 are well-behaved (form a non-singular coordinate system), we can find an explicit map from $T_{\mathbf{p}}\mathbb{R}^2$ to $T_{\mathbf{X}(\mathbf{p})}\mathbb{R}^3$ at a point $\mathbf{p}$ by taking appropriate derivatives of $\mathbf{X}$ as a vector-valued function. Define

$$\mathbf{X}_u(u,v) = \frac{\partial}{\partial u}\mathbf{X} = \begin{bmatrix} \frac{\partial}{\partial u} f^1(u,v) \\ \frac{\partial}{\partial u} f^2(u,v) \\ \frac{\partial}{\partial u} f^3(u,v) \end{bmatrix} \text{ and } \mathbf{X}_v(u,v) = \frac{\partial}{\partial v}\mathbf{X}_{=} \begin{bmatrix} \frac{\partial}{\partial v} f^1(u,v) \\ \frac{\partial}{\partial v} f^2(u,v) \\ \frac{\partial}{\partial v} f^3(u,v) \end{bmatrix}.$$

Now let $\mathbf{v} \in T_{\mathbf{p}}\mathbb{R}^2$ be defined as

$$\mathbf{v} = a\partial_u + b\partial_v,$$

where ∂_u and ∂_v are the basis elements for $T_{\mathbf{p}}\mathbb{R}^2$. (Recall that $\partial_u = \frac{\partial}{\partial u}$ and that derivatives form a basis for the tangent space.) So, in the ∂_u, ∂_v basis,

$$\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then

$$d\mathbf{X} = \mathbf{X}_u du + \mathbf{X}_v dv = \begin{bmatrix} \frac{\partial}{\partial u} f^1(u,v) \\ \frac{\partial}{\partial u} f^2(u,v) \\ \frac{\partial}{\partial u} f^3(u,v) \end{bmatrix} du + \begin{bmatrix} \frac{\partial}{\partial v} f^1(u,v) \\ \frac{\partial}{\partial v} f^2(u,v) \\ \frac{\partial}{\partial v} f^3(u,v) \end{bmatrix} dv, \tag{2.4.4}$$

where du and dv are the basis 1-forms dual to ∂_u and ∂_v. In the du, dv basis for the cotangent space, we have

$$d\mathbf{X} = \begin{bmatrix} \partial_u f^1(u,v) & \partial_v f^1(u,v) \\ \partial_u f^2(u,v) & \partial_v f^2(u,v) \\ \partial_u f^3(u,v) & \partial_v f^3(u,v) \end{bmatrix}. \tag{2.4.5}$$

Then, for $\mathbf{v}$ based at $\mathbf{p} = (u_0, v_0)$,

$$\begin{aligned} d\mathbf{X}(\mathbf{v}) &= (\mathbf{X}_u du + \mathbf{X}_v dv)(a\partial_u + b\partial_v) \\ &= \begin{bmatrix} \partial_u f^1(u,v) & \partial_v f^1(u,v) \\ \partial_u f^2(u,v) & \partial_v f^2(u,v) \\ \partial_u f^3(u,v) & \partial_v f^3(u,v) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \end{aligned} \tag{2.4.6}$$

Remark 2.4.4 The basis ∂_u, ∂_v spans $T_{\mathbf{p}}\mathbb{R}^2$. However, the pair of vectors $\mathbf{X}_u$, $\mathbf{X}_v$ *does not* span $T_{\mathbf{X}(\mathbf{p})}\mathbb{R}^3$. $T_{\mathbf{X}(\mathbf{p})}\mathbb{R}^3$ is a three-dimensonal vector space, so it can only be spanned by three linearly independent vectors. For a given $\mathbf{p}$, $d\mathbf{X}_{\mathbf{p}}$ maps to a plane spanned by $\mathbf{X}_u(\mathbf{p})$ and $\mathbf{X}_v(\mathbf{p})$. This is the *tangent plane* of the surface Σ embedded in $\mathbb{R}^3$.

This may all seem very abstract, but appears in a very common setting.

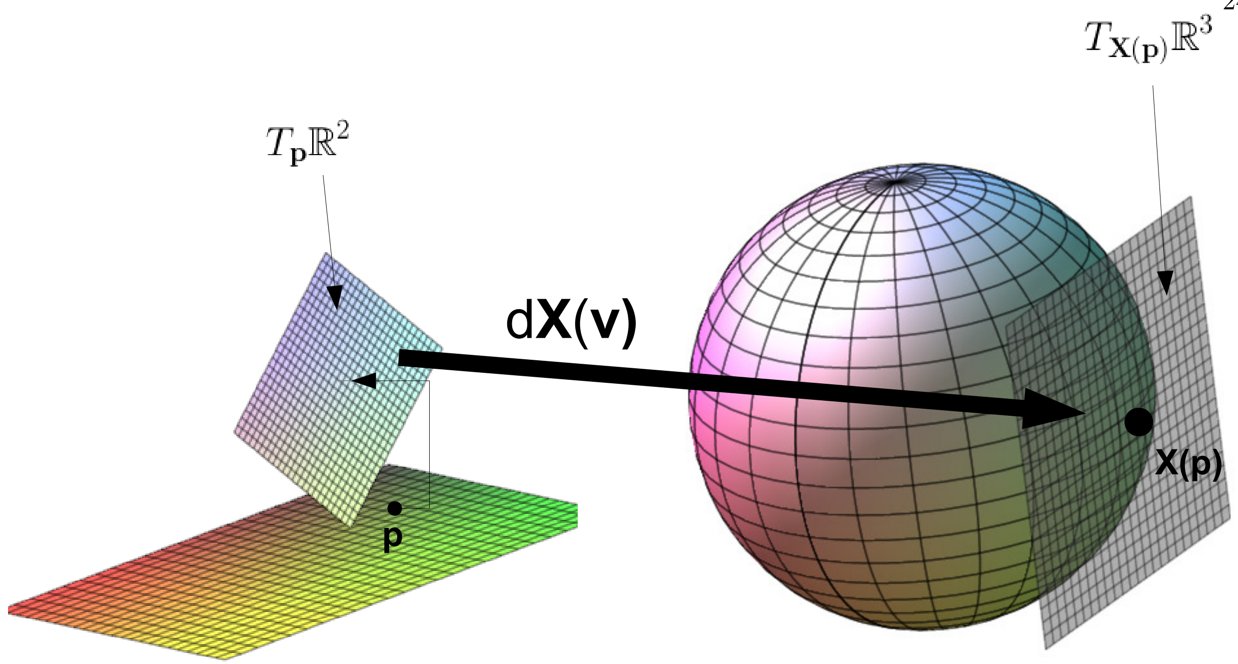

Figure 2.2: $d\mathbf{X}$ maps $T_{\mathbf{p}}R^2$ (left) to the tangent plane of a surface embedded in $\mathbb{R}^3$ (right).

Example 2.4.5 Let

$$\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^2$$

be an invertible function defining a change of coordinates on $\mathbb{R}^2$ defined as

$$\mathbf{X}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix},$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions. Then, in the dx, dy basis for the cotangent space of $\mathbb{R}^2$, $d\mathbf{X}$ is just the Jacobian matrix for a change of variables: [3]

$$d\mathbf{X} = \begin{bmatrix} \partial_x u(x, y) & \partial_y u(x, y) \\ \partial_x v(x, y) & \partial_y v(x, y) \end{bmatrix}.$$

Now let $F : M \to N$ be a map between manifolds. If we define a vector field $\mathbf{v}$ on M, then $dF(\mathbf{v})$ forms a vector field on $F(M) \subseteq N$ defined at each point $F(\mathbf{p}) \in N$ by

$$dF(\mathbf{v})(F(\mathbf{p})) = dF_{\mathbf{p}}(\mathbf{v}(\mathbf{p})) \in T_{F(\mathbf{p})}N. \tag{2.4.7}$$

We often call the vector field $dF(\mathbf{v})$ the *pushforward* of $\mathbf{v}$ and denote it $F_*(\mathbf{v})$.

2.5 Pullbacks

Simply put, a pullback is dual to a pushforward. Given a map between manifolds $F : M \to N$ and a point $p \in M$, a pullback, denoted F^*, maps 1-forms in the cotangent space of N at $F(\mathbf{p})$ to the cotangent space of M based at the point $\mathbf{p} = F^{-1}(F(\mathbf{p}))$. This motivates the following definition.

Definition 2.5.1 Let $F : M \to N$ be a differentiable map, and let $\mathbf{p} \in M$. Then the *pullback* F^* is the unique linear map between cotangent spaces

$$F^*_{F(\mathbf{p})} : T^*_{F(\mathbf{p})}N \to T^*_{\mathbf{p}}M \tag{2.5.1}$$

defined by the following properties:

- If ϕ is a p-form on N, $F^*\phi$ is the p-form defined on M such that

  $$(F^*\phi)(\mathbf{v}_1, ...\mathbf{v}_p) = \phi(F_*(\mathbf{v}_1), ..., F_*(\mathbf{v}_p)), \tag{2.5.2}$$

  where $\mathbf{v}_i$ are vectors in $T_{\mathbf{p}}M$. Recall that F_* is the pushforward.

- For every differentiable function mapping N to the real line, we have

  $$(F^*f) : M \to \mathbb{R} \text{ defined by } (F^*f)(\mathbf{p}) = (f \circ F)(\mathbf{p}), \tag{2.5.3}$$

  where $\mathbf{p} \in M$ [2].

Just like with pushforwards, we can define pullbacks on covector fields by pulling back at each point on N.

Remark 2.5.2 Although covectors in N may be linearly independent, their pullbacks in M may not be. Consider the case where $N = \mathbb{R}^3$ and $M = \mathbb{R}^2$. Now consider the basis covector fields in $\mathbb{R}^3$

$$dx,\ dy,\ \text{and } dz,$$

where x, y, and z are the typical Cartesian coordinates. dx, dy, and dz are linearly independent, indeed orthogonal. However, no matter what map $F : M \to N$ or base point $F(\mathbf{p})$ we choose, the pullbacks

$$F^*(dx),\ F^*(dy),\ \text{and } F^*(dz)$$

will not be linearly independent. The cotangent space in $\mathbb{R}^2$ is too small and no set of three covectors is linearly independent in $\mathbb{R}^2$. We will will make heavy use of this property later.

2.6 Cartan's Lemma

Although we don't need it here, we will use the following result very often in later chapters.

Lemma 2.6.1 Cartan's Lemma:

Suppose that $\eta^1, ..., \eta^n$ are linearly independent 1-forms and that $\phi_1, ..., \phi_n$ are 1-forms such that

$$\phi_i \wedge \eta^i = 0. \tag{2.6.1}$$

Then there exist functions $h_{ij} = h_{ji}$, symmetric in their lower indices, such that

$$\phi_i = h_{ij}\eta^j. \tag{2.6.2}$$

This lemma is really just a reflection of the properties of 1-forms. The proof is just a bit of counting:

Proof:

Suppose that the ϕ_i's and η^i's live on a manifold M of dimension $s \geq n$. We can choose 1-forms $\eta^{n+1}, ..., \eta^s$ so that $\eta^1, ..., \eta^n, \eta^{n+1}, ..., \eta^s$ form a basis for the 1-forms at each point on M. Then, because we have a basis, we can write

$$\phi_i = h_{i\alpha}\eta^\alpha, \ i = 1...n, \ \alpha = 1...s, \tag{2.6.3}$$

where the $h_{i\alpha}$'s are the appropriate coefficients. Substituting equation 2.6.3 into equation 2.6.1 yields the result.

2.7 Affine and Euclidean Geometry

We will primarily be working in affine space. To better understand affine space, it is helpful to compare it to Euclidean space.

2.7.1 Euclidean Geometry

Three-dimensional *Euclidean space* $\mathbb{E}^3$ consists of the vector space $\mathbb{R}^3$, together with a positive definite inner product

$$\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

defined in Cartesian coordinates by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{3} x^i y^i, \tag{2.7.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors with coordinates x^i and y^i respectively. In the language of tensors,

$$\langle \mathbf{x}, \mathbf{y} \rangle = x^i \delta_{ij} y^j, \tag{2.7.2}$$

where δ_{ij} is the Kronecker delta. The inner product has the property that if $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the Cartesian unit vectors, then

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

This is the typical vector dot product, and it defines a way to measure distance—the length of a vector $\mathbf{v}$ is simply $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. Similarly, the distance along a curve is simply the curve's arclength: the length of its tangent vector integrated along it [2].

The *Euclidean group* $E(3)$ is the group of all transformations $\phi : \mathbb{E}^3 \to \mathbb{E}^3$ which preserve the inner product structure; it consists of all transformations of the form

$$\phi(\mathbf{x}) = A\mathbf{x} + \mathbf{b},$$

where $A \in O(3, \mathbb{R})$,[4] the orthogonal group on $\mathbb{R}^3$, and $\mathbf{v} \in \mathbb{E}^3$ [2].

2.7.2 Affine Geometry

Three dimensional *equiaffine space* $\mathbb{A}^3$ (which for convenience we will refer to simply as "affine space") consists of the vector space $\mathbb{R}^3$, together with a nondegenerate volume form

$$dV : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \tag{2.7.3}$$

[4] $O(3, \mathbb{R})$—or the orthogonal group—is the set of all rotations and reflections in 3-dimensional space. It is usually represented by orthogonal 3×3 matrices. Since it is a Lie group, it is endowed with both group and manifold structures. The binary operation is matrix multiplication.

defined by

$$dV(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \det[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3], \tag{2.7.4}$$

where $\mathbf{v}_i$ are vectors and $[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3]$ is the matrix formed by taking the vectors as the columns of a matrix. det is the typical matrix determinant. The volume form has the property that

$$dV(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e_3}) = 1.$$

Like the inner product of Euclidean geometry, the affine volume form is multilinear [2]. In other words, it is linear in each of its indices. i.e., if $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{b} \in \mathbb{R}^3$ and $a \in \mathbb{R}$,

$$\begin{aligned} dV(a\mathbf{x}+\mathbf{b}, \mathbf{y}, \mathbf{z}) &= adV(\mathbf{x}, \mathbf{y}, \mathbf{z}) + dV(\mathbf{b}, \mathbf{y}, \mathbf{z}) \\ dV(\mathbf{x}, a\mathbf{y}+\mathbf{b}, \mathbf{z}) &= adV(\mathbf{x}, \mathbf{y}, \mathbf{z}) + dV(\mathbf{x}, \mathbf{b}, \mathbf{z}) \\ dV(\mathbf{x}, \mathbf{y}, a\mathbf{z}+\mathbf{b}) &= adV(\mathbf{x}, \mathbf{y}, \mathbf{z}) + dV(\mathbf{x}, \mathbf{y}, \mathbf{b}) \end{aligned} \tag{2.7.5}$$

Unlike the Euclidean inner product, the affine volume form form is skew-symmetric [2]. In other words,

$$\begin{aligned} dV(\mathbf{y}, \mathbf{x}, \mathbf{z}) &= -dV(\mathbf{x}, \mathbf{y}, \mathbf{z}) \\ dV(\mathbf{x}, \mathbf{z}, \mathbf{y}) &= -dV(\mathbf{x}, \mathbf{y}, \mathbf{z}). \end{aligned} \tag{2.7.6}$$

The *equiaffine group* $A(3)$ is the group of all transformations $\phi : \mathbb{E}^3 \to \mathbb{E}^3$ which preserve the volume form; it consists of transformations of the form

$$\phi(\mathbf{x}) = A\mathbf{x} + \mathbf{v},$$

where $A \in SL(3, \mathbb{R})$ and $b \in \mathbb{A}^3$.[5] Elements of $A(3)$ are also called *affine transformations* or *unimodular transformations* [2].

The volume form $dV(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ measures the signed volume of the parallelepiped spanned by the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. However, unlike the Euclidean inner product, the volume form says nothing about the lengths of individual vectors. Indeed, transformations that preserve the affine structure—or affine transformations—don't necessarily preserve Euclidean distance at all. So long as two parallelepipeds have the same volume, they are considered equivalent objects [2].

[5] $SL(3, \mathbb{R})$—or the special linear group—is the set of all linear transformations that preserve volume. It can be represented as the set of all 3×3 matrices with a determinant of 1. Since it is a Lie group, it is endowed with both group and manifold structures. The binary operation is matrix multiplication.

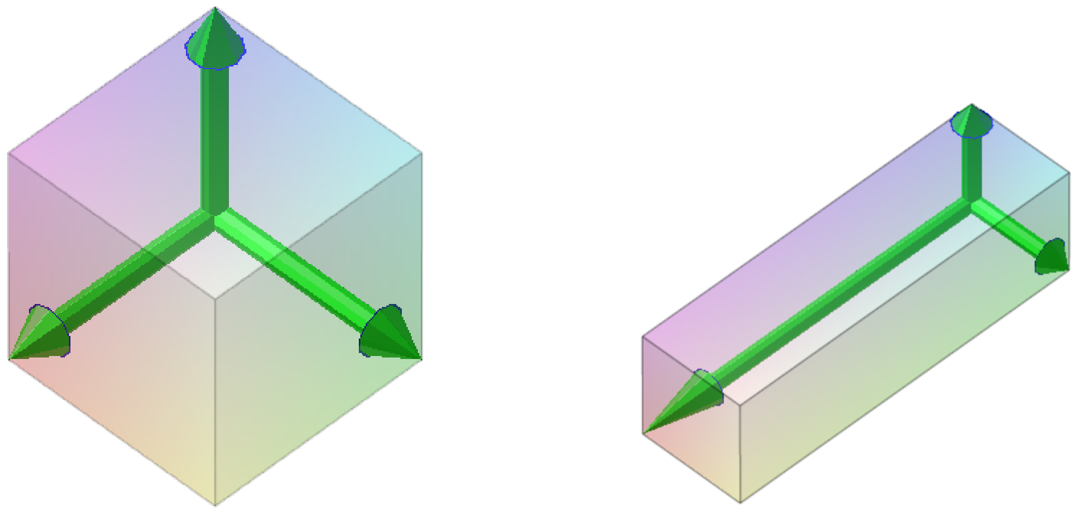

Figure 2.3: These two paralellepipeds are considered the same in equiaffine space, despite the different lengths of the individual vectors that define them.

2.8 An Aside on Euclidean Gauss Curvature

To conclude this chapter, we will briefly discuss Euclidean Gauss curvature from an intuitive point of view. This idea is important in chapter 4.

Let $U \subseteq \mathbb{R}^2$ and let Σ be a surface embedded in $\mathbb{E}^3$ parametrized by

$$\mathbf{X} : U \to \mathbb{E}^3.$$

For every point $\mathbf{p} \in U$, define $\mathbf{N}(\mathbf{p})$ to be the unit vector normal to Σ at a given point $\mathbf{X}(\mathbf{p})$. Now, let

$$\alpha : I \to \mathbb{E}^3$$

be a curve embedded in Σ. If we move $\mathbf{N}$ along α as $\mathbf{N}(\alpha(t))$ and enforce that it stay normal to Σ, then the *normal curvature* of Σ along a path α is the projection of $\frac{d\mathbf{N}(\alpha(t))}{dt}$ onto the tangent vector $\alpha'(t)$:

$$k(\alpha') = -\left(\frac{d\mathbf{N}}{dt}\right) \cdot \alpha'. \tag{2.8.1}$$

Normal curvature can be positive or negative.

Figure 2.4 shows how normal curvature works. If we transport our unit vector (red) along either path (green), it changes direction. If it moves towards α', the normal curvature is negative. If it moves away,

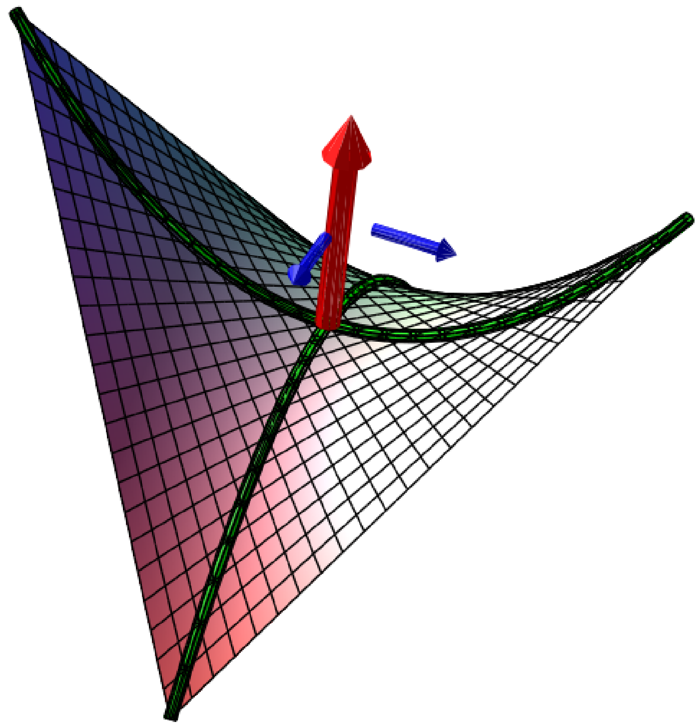

Figure 2.4: Normal curvature along a surface. If we parallel transport our unit vector (red) along either path (green), it changes direction. If it moves towards α', the normal curvature is negative. If it moves away, the normal curvature is positive.

the normal curvature is positive.[6] The amount the direction changes determines the overall magnitude of normal curvature. We can define normal curvature at a point in a direction by differentiating our unit normal vector along any curve initially traveling in the appropriate direction. In figure 2.4, the initial directions are the blue arrows.

We call the maximum normal curvature at a given point κ_1 and the minimum normal curvature at a given point κ_2. Then we define Euclidean Gauss curvature as their product:

$$K_{\mathbb{E}} = \kappa_1 \kappa_2. \tag{2.8.2}$$

Euclidean Gauss curvature can be positive or negative. It's positive if the normal curvature is the same sign in all directions at a point. It's negative if the maximum normal curvature is positive and the minimum

[6] The normal curvature is only well defined up to an overall sign. However, the relative sign between normal curvatures is well defined.

normal curvature is negative. Positive Euclidean Gauss curvature surfaces look locally like spheres. Negative Euclidean Gauss curvature surfaces look locally like saddle points.

Because normal curvature varies continuously at a point, if a surface has negative Euclidean Gauss curvature, there are two linearly independent curves passing through each point of the surface along which the normal curvature is zero. We call these curves *asymptotic curves* .

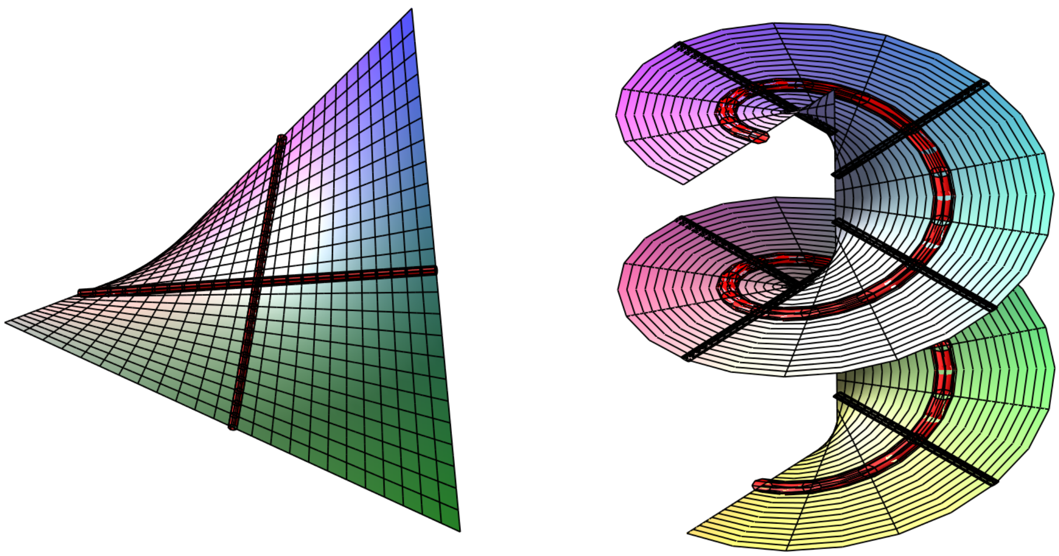

Figure 2.5: The asymptotic curves for a saddle point (left) are straight lines and the asymptotic curves for a helicoid (right) are helices and straight lines.

Chapter 3

General Affine Theory

Now that we have some background, we can understand the general theory of surfaces in equiaffine space. Once we understand this general theory, we can move on to characterizing affine minimal flat surfaces. The general theory is developed here using the method of moving frames. This treatment comes from [2].

3.1 A Unimodular Frame

Let U be an open set in $\mathbb{R}^2$ and let $\mathbf{X} : U \to \mathbb{A}^3$ be the parametrization of a surface $\Sigma = \mathbf{X}(U)$. We want to generalize the Frenet frame from section 1.4.1. To that end, we define a *unimodular frame.*

Definition 3.1.1 At each point $\mathbf{p} = (u, v) \in U$,[1] we choose a set of three linearly independent vectors in $T_{\mathbf{X}(\mathbf{p})}\mathbb{A}^3$, $\mathbf{e}_1(\mathbf{p}) := \mathbf{e}_1(\mathbf{X}(\mathbf{p})), \mathbf{e}_2(\mathbf{p}) := \mathbf{e}_2(\mathbf{X}(\mathbf{p})), \mathbf{e}_3(\mathbf{p}) = \mathbf{e}_3(\mathbf{X}(\mathbf{p}))$, such that

$$\det\left([\mathbf{e}_1(\mathbf{p}), \mathbf{e}_2(\mathbf{p}), \mathbf{e}_3(\mathbf{p})]\right) = 1 \ \forall \ \mathbf{p} \in U, \tag{3.1.1}$$

where $[\mathbf{e}_1(\mathbf{p}), \mathbf{e}_2(\mathbf{p}), \mathbf{e}_3(\mathbf{p})]$ is the matrix formed by taking each $\mathbf{e}_i$ as a column. This collection of vectors is called a *unimodular frame.* A single triple of such vectors based at a given point (as opposed to three vector *fields*) is called a *unimodular basis* for the tangent space at that point.

3.1.1 The Maurer-Cartan Forms

The vectors $\mathbf{e}_i$, $i \in \{1, 2, 3\}$ form a basis for the tangent space $T_{\mathbf{X}(\mathbf{p})}\mathbb{A}^3$. So, we can use them to represent other vectors. Critically, we can write $d\mathbf{X}$ as a linear combination of these vectors. We call the

[1] From now on, unless otherwise stated, $\mathbf{p}$ stands for an arbitrary point so constructed.

coefficients ω^i such that

$$d\mathbf{X} = \mathbf{e}_i \omega^i. \tag{3.1.2}$$

Each ω^i is a scalar-valued 1-form defined in the cotangent space of U based at $\mathbf{p}$ in $T^*_{\mathbf{p}}U$. The ω^i's describe how the surface changes in the directions of the $\mathbf{e}_i$'s. They're also called the *dual forms* of the frame because each ω^i is dual to a frame vector:

$$\omega^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}. \tag{3.1.3}$$

Since *every* vector can be written as a linear combination of the $\mathbf{e}_i$'s, the derivatives of the $\mathbf{e}_i$'s themselves can be decomposed in this way too.[2] Thus, we define another set of scalar-valued 1-forms. We call these ω^j_i such that

$$d\mathbf{e}_i = \mathbf{e}_j \omega^j_i. \tag{3.1.4}$$

Conceptually, ω^j_i describes how the $\mathbf{e}_i$ vector changes in the $\mathbf{e}^{th}_j$ direction. Together the ω^i's and the ω^i_j's are called the *Maurer-Cartan forms.*

We can represent these 1-forms in matrix notation in the following way. Let

$$A = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \tag{3.1.5}$$

be the matrix in $SL(3, \mathbb{R})$ defined by taking each $\mathbf{e}_i$ as a column. Then

$$d\mathbf{X} = A \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} \tag{3.1.6}$$

and thus

$$\begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{bmatrix} = A^{-1} d\mathbf{X} = A^{-1} \begin{bmatrix} dX^1 \\ dX^2 \\ dX^3 \end{bmatrix}. \tag{3.1.7}$$

[2] Conceptually this is very much the same as the procedure in section 1.4.1.

Similarly, for the ω^i_j coefficients,

$$\begin{aligned} dA &= [d\mathbf{e}_1, d\mathbf{e}_2, d\mathbf{e}_3] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix} \\ \Rightarrow dA &= A \begin{bmatrix} \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix}. \qquad (3.1.8) \end{aligned}$$

$$(3.1.9)$$

So,

$$\begin{bmatrix} \omega^1_1 & \omega^1_2 & \omega^1_3 \\ \omega^2_1 & \omega^2_2 & \omega^2_3 \\ \omega^3_1 & \omega^3_2 & \omega^3_3 \end{bmatrix} = A^{-1} dA. \qquad (3.1.10)$$

Remark 3.1.2 For each point $\mathbf{p} \in U$, the Maurer-Cartan forms map vectors tangent to U to the real line.

3.1.2 The Structure Equations

The generalization of the Frenet frame from section 1.4.1 is a linearly independent basis defined at each point on Σ. As we will see, we can choose a much better basis for our surface than just a unimodular one. For now, however, we need to figure out how to generate the analog of the Frenet equations. If we differentiate equations 3.1.2 and 3.1.8, we find that

$$\begin{aligned} d\omega^i &= -\omega^i_j \wedge \omega^j \qquad (3.1.11) \\ d\omega^i_j &= -\omega^i_k \wedge \omega^k_j. \end{aligned}$$

We call these the *Cartan Structure Equations*, or the *structure equations* for short. The structure equations will play an important role for us as we try to calculate the properties of affine surfaces.

3.2 Frame Adaptation

For a given surface Σ and a given embedding $\mathbf{X} : U \to \mathbb{A}^3$, there are many choices of unimodular frame. We want to choose a frame that is geometrically natural. To get there, we'll slowly make choices about the frame we want. These choices will give us more information about our surface, which will inform the next choice. We call this process *frame adaptation.*

By differentiating (3.1.1), we can discover another relationship between the Maurer-Cartan forms. First, let $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ be a constant standard basis in $T\Sigma$.

$$
\begin{aligned}
1 &= \det[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \\
\Rightarrow \underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3 &= \det[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\
\Rightarrow d\,(\underline{e}_1 \wedge \underline{e}_2 \wedge \underline{e}_3) &= d\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \\
\Rightarrow 0 &= d\,(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \\
&= d\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge d\mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge d\mathbf{e}_3 \\
&= \mathbf{e}_i \omega^i{}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_i \omega^i{}_2 \wedge \mathbf{e}_3 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_i \omega^i{}_3 \\
&= \left(\omega^1{}_1 + \omega^2{}_2 + \omega^3{}_3\right) \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\
\Rightarrow 0 &= \omega^1{}_1 + \omega^2{}_2 + \omega^3{}_3 = \omega^i{}_i \qquad (3.2.1)
\end{aligned}
$$

This is called the *trace condition.*

Remark 3.2.1 Like the wedge product, the determinant is skew-symmetric, and this is no accident. The determinant of a matrix can be defined in terms of the wedge products of the covectors of the matrix.

3.2.1 0-Adaptation

We now begin the process of choosing a suitable frame. The relative angles between vectors aren't preserved under affine transformation, so choosing a frame where all three vectors are orthogonal isn't very helpful. Instead, we'll do the next best thing. We choose a frame where

$$[\mathbf{e}_1(u,v), \mathbf{e}_2(u,v)] \ \text{span}\ T_{\mathbf{X}(u,v)}\Sigma. \qquad (3.2.2)$$

This property is preserved by affine transformation. Now we can ask how the Maurer-Cartan forms are affected by this constraint.

If $\mathbf{p} = (p^1, p^2) \in U$, then $d\mathbf{X}_\mathbf{p} = d\mathbf{X}(p^1, p^2)$ is a linear map from $T_\mathbf{p}U$ to $T_{\mathbf{X}(\mathbf{p})}\Sigma$ such that

$$d\mathbf{X}_\mathbf{p}(\mathbf{v}) = \mathbf{e}_i(\mathbf{p})\omega^i(\mathbf{v}) \in T_{\mathbf{X}(\mathbf{u})}\Sigma,$$

where $\mathbf{v} \in T_\mathbf{p}U$. Since $T_{\mathbf{X}(\mathbf{p})}\Sigma$ is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, the $\mathbf{e}_3$ term must be zero for all input vectors. Thus,

$$\omega^3(\mathbf{v}) = 0. \tag{3.2.3}$$

Furthermore, when we pull back, we find that ω^1 and ω^2 form a basis for the 1-forms on U. Now we use the Cartan structure equations (equation 3.1.11). Remember, "when in doubt, differentiate!"

Let's differentiate the equation $\omega^3 = 0$:

$$\begin{aligned} 0 &= d\omega^3 \\ &= -\omega^3_1 \wedge \omega^1 - \omega^3_2 \wedge \omega^2 \end{aligned}$$

Then by Cartan's Lemma (see section 2.6), there exists a symmetric matrix $H = [h_{ij}]$ such that

$$\begin{bmatrix} \omega^3_1 \\ \omega^3_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}. \tag{3.2.4}$$

We'll use this information later to inform our choice of frame.

3.2.2 1-Adaptation

There are still many choices for 0-adapted frames. Let's see how we can transform between 0-adapted frames. Let $(\mathbf{e}_1(u,b), \mathbf{e}_2(u,v), \mathbf{e}_3(u,v))$ be any 0-adapted frame with Maurer-Cartan forms ω^i and ω^i_j. Let $(\tilde{\mathbf{e}}_1(u,v), \tilde{\mathbf{e}}_2(u,v), \tilde{\mathbf{e}}_3(u,b))$ be another adapted frame. Because of the conditions we've imposed, we know that at each point on Σ,

$$\operatorname{span}(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}) = \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2)$$

and

$$[\tilde{\mathbf{e}}_1\ \tilde{\mathbf{e}}_2\ \tilde{\mathbf{e}}_3] = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3] \begin{bmatrix} b_{11} & b_{12} & r_1 \\ b_{21} & b_{22} & r_2 \\ 0 & 0 & (\det B)^{-1} \end{bmatrix},$$

where $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ is an element of $GL(2)$, the set of all invertible 2×2 matrices. Let the Maurer-Cartan forms of the new frame be $\tilde{\omega}^i$ and $\tilde{\omega}_i^j$. Then equation 3.2.4 becomes

$$\begin{bmatrix} \tilde{\omega}_1^3 \\ \tilde{\omega}_2^3 \end{bmatrix} = \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\omega}^1 \\ \tilde{\omega}^2 \end{bmatrix}, \tag{3.2.5}$$

where $\tilde{h}_{ij}$ are the coefficients given by Cartan's Lemma. Some careful calculation tells us that

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_{22} \end{bmatrix} = (\det B)^t B \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} B. \tag{3.2.6}$$

This means that

$$\det([\tilde{h}_{ij}]) = (\det B)^4 \det([h_{ij}]). \tag{3.2.7}$$

So the the sign of the determinant of the matrix $[h_{ij}]$ is fixed. If $\det([h_{ij}]) > 0$, we say that the surface Σ is *elliptic*. If $\det([h_{ij}]) < 0$, we say that Σ is *hyperbolic*.[3] We're only interested in the hyperbolic case, so we will assume from this point on that $\det([h_{ij}]) < 0$.

Since $[h_{ij}]$ is a 2×2 matrix with real eigenvalues, we can choose our frame so that

$$[h_{ij}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.2.8}$$

We call a frame with this property *1-adapted*.

Now that our frame is 1-adapted, equation 3.2.4 tells us that

$$\omega_1^3 = \omega^2 \text{ and } \omega_2^3 = \omega^1. \tag{3.2.9}$$

This lets us define a notion of distance, which we call the *affine first fundamental form*:

$$\begin{aligned} I &:= \omega_1^3\omega^1 + \omega_2^3\omega^2 \\ &= 2\omega^1 \circ \omega^2, \end{aligned} \tag{3.2.10}$$

which is independent of the choice of 1-adapted frame.

[3] A surface can have $\det([h_{ij}]) = 0$. However this means the surface is degenerate in some way, and difficult to handle. We call these surfaces *degenerate*, and we won't deal with any here.

The first fundamental form, also called the *affine metric*, gives us a notion of distance. It maps a pair of vectors to the real line (remember that ω^1 and ω^2 are 1-forms) and it's invariant under affine transformation. However, it is not like the Euclidean notion of distance at all. First of all, because our surface is hyperbolic, it can be negative—there's no reason that given vectors $\mathbf{v}, \mathbf{w}$ in $T_{\mathbf{p}}\Sigma$, $(\omega^1 \circ \omega^2)(\mathbf{v}, \mathbf{w})$ must be positive. In this way, the affine metric is much more similar to a Lorentzian metric from general relativity.

The affine first fundamental form also depends on second-order derivatives while the Euclidean first fundamental form only depends on first-order derivatives. This makes the affine first fundamental form much more similar to Euclidean *curvature*, not distance. However, this is what passes for distance in affine space. We make do.

Remark 3.2.2 Elliptic affine surfaces have strictly positive affine first fundamental form. This means that their affine metric is closer to a Euclidean metric than a Lorentzian one. However, unlike a Euclidean metric, the affine metric for elliptic surfaces still depends on higher-order derivatives.

3.2.3 2-Adaptation

Now we play the same game as we did for a 0-adapted frame. There are many choices for 1-adapted frames. If

$$(\mathbf{e}_1(u,v), \mathbf{e}_2(u,v), \mathbf{e}_3(u,v))$$

is one 1-adapted frame, then any other 1-adapted frame $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ must be of the form

$$[\mathbf{e}_1(u,v)\ \mathbf{e}_2(u,v)\ \mathbf{e}_3(u,v))] = [\tilde{\mathbf{e}}_1(u,v)\ \tilde{\mathbf{e}}_2(u,v)\ \tilde{\mathbf{e}}_3(u,v)] \begin{bmatrix} e^{\theta} & 0 & r_1 \\ 0 & e^{-\theta} & r_2 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.2.11}$$

where θ, r_1, and r_2 are arbitrary functions of u and v. The new frame has an associated Maurer-Cartan form $\tilde{\omega}_3^3$, which must be equal to

$$\tilde{\omega}_3^3 = \omega_3^3 + r_2\omega^1 + r_1\omega^2. \tag{3.2.12}$$

Since r_1 and r_2 are arbitrary, we can choose a 1-adapted frame at each u and v such that ω_3^3 is identically zero. We call such a frame *2-adapted*.

The condition

$$\omega_3^3 = 0 \tag{3.2.13}$$

gives us something to differentiate. And remember: when in doubt, differentiate! The structure equations (equation 3.1.11) tell us that

$$\begin{aligned} 0 &= d\omega_3^3 = -\omega_k^3 \wedge \omega_3^k \\ &= -\omega_1^3 \wedge \omega_3^1 - \omega_2^3 \wedge \omega_3^2 - \omega_3^3 \wedge \omega_3^3 \\ \Rightarrow 0 &= -\omega_1^3 \wedge \omega_3^1 - \omega_2^3 \wedge \omega_3^2 \text{ because } \wedge \text{ is antisymmetric} \\ &= \omega_3^1 \wedge \omega^2 + \omega_3^2 \wedge \omega^1. \end{aligned} \tag{3.2.14}$$

We can then use Cartan's Lemma (see section 2.6) to say that

$$\begin{bmatrix} \omega_3^1 \\ \omega_3^2 \end{bmatrix} = \begin{bmatrix} l & l_{12} \\ l_{21} & l \end{bmatrix} \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}, \tag{3.2.15}$$

where l, l_{12}, and l_{21} are functions of u and v. Given this relationship, we can define an *affine second fundamental form* on the tangent space of Σ based at a given point:

$$\begin{aligned} II &= \omega_3^1\omega_1^3 + \omega_3^2\omega_2^3 \\ &= \omega_3^1\omega^2 + \omega_3^2\omega^2 \\ &= l_{21}(\omega^1)^2 + 2l\omega^1\omega^2 + l_{12}(\omega^2)^2, \end{aligned} \tag{3.2.16}$$

which is unique and invariant up to choice of 2-adapted frame. Like the affine first fundamental form (equation 3.2.10), the affine second fundamental form is a quadratic 2-form: It eats two vectors and returns a real number.

In Euclidean space, the *Euclidean second fundamental form* measures the curvature of a surface at a given point. Although the affine second fundamental form depends on higher-order derivatives than the Euclidean form, it is the generalization into affine space of this notion of curvature.[4] Importantly, we define

[4] As with the affine first fundamental form, you make do with what you have.

the affine Gauss curvature,

$$K_{\mathbb{A}} = \det \left(\begin{bmatrix} l & l_{12} \\ l_{21} & l \end{bmatrix} \right) = l_{12}l_{21} - l^2, \tag{3.2.17}$$

and affine mean curvature,

$$L_{\mathbb{A}} = \operatorname{trace} \left(\begin{bmatrix} l & l_{12} \\ l_{21} & l \end{bmatrix} \right) = 2l. \tag{3.2.18}$$

A surface with zero affine Gauss curvature is called *affine flat*. A surface with zero affine mean curvature is called *affine minimal*. Our main result is the characterization of affine flat, affine minimal surfaces. So these definitions are particularly important to us.

We could continue to play our frame adaptation game some more and choose a unique frame. However, the results in the next section only require the affine first and second fundamental forms, so we'll stop here.

Chapter 4

Characterization

Now we will present the main result, where we characterize hyperbolic affine flat, affine minimal surfaces. This means that we assume our surface has constant negative Euclidean Gauss curvature and zero affine Gauss and mean curvatures. For comparison, the only surface with zero Euclidean Gauss and mean curvatures is the plane. However, as we shall see, there are many examples in equiaffine space.

4.1 Frame Adaptation.

We will use the same technique we did when we developed the general theory. We'll choose a frame based in the tangent space of an appropriate surface at each point in a geometrically natural way. Let $U \subseteq \mathbb{R}^2$ and let $\Sigma \subseteq \mathbb{A}^3$ be our affine flat affine minimal surface. We parametrize Σ with the function $\mathbf{X}(u, v)$ such that

$$\mathbf{X} : U \to \mathbb{A}^3, \ U \subseteq \mathbb{R}^2.$$

Because Σ is affine flat and affine minimal,

$$K_{\mathbb{A}} = L_{\mathbb{A}} = 0$$

at every point $\mathbf{p} \in U$.

For each point $\mathbf{p} \in U$, we choose three linearly independent vectors, $\mathbf{e}_i(\mathbf{p}) = \mathbf{e}_i(\mathbf{X}(\mathbf{p})), \ i \in \{1, 2, 3\}$ in $T_{\mathbf{X}(\mathbf{p})}\mathbb{A}^3$ such that the volume form associated with the vectors evaluates to unity:

$$V(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \det[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1. \qquad (4.1.1)$$

These vectors define an *affine frame field.*

4.1.1 Initial Adaptation

If Σ is hyperbolic, then not only does it meet the conditions outlined in section 3.2.2, but it also has constant negative Euclidean Gauss curvature. A Euclidean surface with constant negative Gauss curvature looks locally like a saddle point and has two asymptotic curves.

We can choose the parametrization of our surface such that the u, v-parameter curves are asymptotic curves. Then we choose our frame such that $\mathbf{e}_1 = \mathbf{X}_u$ and $\mathbf{e}_2 = \mathbf{X}_v$. Thus, $\mathbf{e}_1$ and $\mathbf{e}_2$ span the tangent plane of $\mathbf{X}$ and we have that

$$\begin{aligned} d\mathbf{X} &= \mathbf{X}_u du + \mathbf{X}_v dv \\ &= \mathbf{e}_1 du + \mathbf{e}_2 dv \\ \Rightarrow \omega^1 &= du,\ \omega^2 = dv,\ \text{and } \omega^3 = 0. \end{aligned} \tag{4.1.2}$$

At this point, so long as we enforce that our frame be unimodular, we now have a zero adapted frame.

4.1.2 Metric Structure

By the first structure equation,

$$\begin{aligned} \omega^3 &= 0 \\ \Leftrightarrow d\omega^3 &= -{\omega^3}_1 \wedge \omega^1 - {\omega^3}_2 \wedge \omega^2 = 0. \end{aligned}$$

By Cartan's lemma, this gives us that

$$\begin{aligned} {\omega^3}_1 &= h_{11}\omega^1 + h_{12}\omega^2 \\ \texttt{and } {\omega^3}_2 &= h_{12}\omega^1 + h_{22}\omega^2. \end{aligned}$$

We can think of the matrix $[h_{ij}]$ as a list of normal curvatures at a given point in the $u - v-$basis. However, because $\mathbf{e}_1$ and $\mathbf{e}_2$ are asymptotic coordinates, the normal curvature in those directions is zero. Thus,

$$h_{11} = h_{22} = 0$$

and

$$\omega^3{}_1 = h_{12}\omega^2 = h_{12}dv \tag{4.1.3}$$

$$\omega^3{}_2 = h_{12}\omega^2 = h_{12}du \tag{4.1.4}$$

In section 3.2.2, we chose our 1-adapted frame to normalize the affine first fundamental form. However, if we're willing to accept an overall normalization factor, we can define an equally invariant object with a bit less information:

$$\begin{aligned} I_{aff} &= |\det(h)|^{-1/4}(\omega^3{}_1\omega^1 + \omega^3{}_2\omega^2) \\ &= 2\sqrt{h_{12}}dudv. \end{aligned} \tag{4.1.5}$$

Since we now have an affine metric, we can then use Gauss's formula to calculate the affine analog of Euclidean Gauss curvature [3]:

$$K_{\mathbb{A}} = -\frac{1}{F}\frac{\partial^2 \log F}{\partial u \partial v}, \tag{4.1.6}$$

where $F = \sqrt{h_{12}}$. We call this the *affine Gauss curvature.* The condition that Σ is affine flat means that

$$\begin{aligned} K_{\mathbb{A}} &= 0 = -\frac{1}{f}\frac{\partial^2 \log F}{\partial u \partial v} \\ \Rightarrow F &= e^{f_1(u)+f_2(v)} = F_1(u)F_1(v) \\ \Rightarrow I_{aff} &= 2F_1(u)duF_2(v)dv, \end{aligned}$$

for some F_1 and F_2. We can then rescale our u and v parameter curves such that $d\tilde{u} = F_1(u)du$ and $d\tilde{v} = F_2(v)dv$. From now on, we will call $d\tilde{u}$ du and $d\tilde{v}$ dv. We also adjust our frame slightly so that $\mathbf{e}_1 = \frac{\partial}{\partial \tilde{u}}$ and $\mathbf{e}_2 = \frac{\partial}{\partial \tilde{v}}$. Thus, we have $F = \sqrt{h_{12}} = 1$. In this frame,

$$\omega_1^3 = \omega^2 \text{ and } \omega_2^1 = \omega^1.$$

So, our frame is now *1-adapted.* Now, after rescaling, we get that

$$I_{aff} = 2dudv. \tag{4.1.7}$$

Like in section 3.2.2, we can also choose an $\mathbf{e}_3$ such that $\omega^3{}_3 = 0$. Such a frame is *2-adapted.* A two adapted frame gives us functions l, l_{12} and l_{21} such that

$$\omega^1{}_3 = ldu + l_{12}dv \ \texttt{and} \ \omega^2{}_3 = l_{21} + ldv.$$

By the general theory (equation 3.2.18), l is proportional to $L_{\mathbb{A}}$, the affine mean curvature. Thus, since, Σ is affine minimal, $L_{\mathbb{A}} = 0$ and

$$\omega^1{}_3 = l_{12}dv \text{ and } \omega^2{}_3 = l_{21}du. \tag{4.1.8}$$

Since $\omega^1 = du$ and $\omega^2 = dv$ and $d(df)$ for any scalar function f,

$$\begin{aligned} 0 &= d\omega^1 \\ &= -\omega^1{}_1 \wedge du - \omega^1{}_2 \wedge dv \\ \text{and } 0 &= d\omega^2 \\ &= -\omega^2{}_1 \wedge du - \omega^2{}_2 \wedge dv. \end{aligned}$$

Thus, by Cartan's lemma, there exist functions h_1, h_2, h_3, and h_4 such that

$$\begin{aligned} \omega^1{}_1 &= h_1 du + h_2 dv \\ \omega^2{}_2 &= -h_1 du - h_2 dv \\ \omega^1{}_2 &= h_2 du + h_3 dv \\ \text{and } \omega^2{}_1 &= h_4 du - h_1 dv. \end{aligned}$$

Now, for similar reasons,

$$\begin{aligned} 0 &= d\omega^3{}_1 \\ &= -dv \wedge (h_1 du + h_2 dv) - du \wedge (h_4 du - h_1 dv) \\ &= h_1 du \wedge dv. \end{aligned}$$

So h_1 must be zero. Similarly,

$$\begin{aligned} 0 &= d\omega^3{}_2 \\ &= -dv \wedge (h_2 du + h_3 dv) + du \wedge (h_2 dv) \\ &= 2h_2 du \wedge dv. \end{aligned}$$

So h_2 must be zero.

We already know that

$$
\begin{aligned}
0 &= \omega^1{}_1 \\
\Rightarrow 0 &= d\omega^1{}_1 \\
&= h_3 h_4 du \wedge dv
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \omega^2{}_2 \\
\Rightarrow 0 &= d\omega^2{}_2 \\
&= -h_3 h_4 du \wedge dv.
\end{aligned}
$$

So either h_3 or h_4 must be zero. Without loss of generality, let $h_4 = 0$. Now, $\omega^2{}_1$ is zero, so

$$
\begin{aligned}
0 &= \omega^2{}_1 \\
\Rightarrow 0 &= d\omega^2{}_1 \\
&= -l_{21} du \wedge dv \\
\Rightarrow l_{21} &= 0. \qquad (4.1.9)
\end{aligned}
$$

Now we need to look at the structure equation for $\omega^1{}_2$. Since it's not necessarily exact, we compare its exterior derivative to the result given by the structure equations.

$$
\begin{aligned}
d\omega^1{}_2 &= -\omega^1_1 \wedge \omega^1{}_1 - \omega^1{}_2 \wedge \omega^2{}_2 - \omega^1{}_3 \wedge \omega^3{}_2 \\
\Rightarrow dh_3 \wedge dv &= l_{12} du \wedge dv \\
\Rightarrow \frac{\partial h_3}{\partial u} &= l_{12}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d\omega^1{}_3 &= -\omega^1{}_1 \wedge \omega^1{}_3 - \omega^1{}_2 \wedge \omega^2{}_3 - \omega^1{}_3 \wedge \omega^3{}_3 \\
\Rightarrow dl_{12} \wedge dv &= 0 \\
\Rightarrow l_{12} &= l_{12}(v).
\end{aligned}
$$

Let $l(v) = l_{12}(v)$. Then $h_3(u,v) = ul(v) + f(v)$ for some function $f(v)$. We have all the Maurer-Cartan forms now, so we can state the following important result.

Theorem 4.1.1 Let Σ be a hyperbolic affine surface with affine Gauss and mean curvatures equal to zero. Then there exist local coordinates (u,v) on Σ and an affine frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ along Σ with $\mathbf{e}_1 = \frac{\partial}{\partial u}$, $\mathbf{e}_2 = \frac{\partial}{\partial v}$, and with associated Maurer-Cartan forms:

$$\omega^1 = du \qquad \omega^2 = dv \qquad \omega^3 = 0 \tag{4.1.10}$$

$$\omega^1{}_1 = 0 \quad \omega^1{}_2 = (ul(v) + f(v))dv \quad \omega^1{}_3 = l(v)dv \tag{4.1.11}$$

$$\omega^2{}_1 = 0 \qquad \omega^2{}_2 = 0 \qquad \omega^2{}_3 = 0 \tag{4.1.12}$$

$$\omega^3{}_1 = dv \qquad \omega^3{}_2 = du \qquad \omega^3{}_3 = 0, \tag{4.1.13}$$

where $l(v)$ and $f(v)$ are some functions.

We're ready to start explicitly solving some systems!

4.2 Integration

The Maurer-Cartan forms generate a unique compatible, overdetermined PDE system for the function $\mathbf{X}(u,v)$.

$$d\mathbf{X} = \mathbf{e}_1 du + \mathbf{e}_2 dv \tag{4.2.1}$$

$$d\mathbf{e}_1 = \mathbf{e}_3 dv \tag{4.2.2}$$

$$d\mathbf{e}_2 = \mathbf{e}_1(ul(v) + f(v))dv + \mathbf{e}_3 du \tag{4.2.3}$$

$$d\mathbf{e}_3 = \mathbf{e}_1 l(v)dv. \tag{4.2.4}$$

We have a PDE system uniquely determined (up to integration constants and affine group action) by $l(v)$ and $f(v)$. Because the system is compatible, we know that we can write it as a system of ODEs. Thus, ODE existence/uniqueness theorems give us the following result.

Theorem 4.2.1 Let $U \subseteq \mathbb{R}^2$ and let let $l(v)$ and $f(v)$ be any continuous, integrable real-valued functions on U. Then for any given $(u, v) \in U$, there exists a neighborhood $V \subseteq U$ of (u, v) and an affine minimal, affine flat surface $\mathbf{X} : V \to \mathbb{A}^3$ uniquely determined up to affine transformation by $l(v)$ and $f(v)$.

This is about as nice as it gets. $\mathbf{e}_1$ and $\mathbf{e}_3$ are independent of $\mathbf{e}_2$ and of u, meaning they're functions of v alone. So we can solve for those first. If we combine them, we get that

$$\frac{\partial^2 \mathbf{e}_1}{\partial v^2} = \mathbf{e}_1 l(v) \;\; \texttt{and} \;\; \frac{\partial \mathbf{e}_1}{\partial v} = \mathbf{e}_3. \tag{4.2.5}$$

So, we just need to solve the second order ODE for $\mathbf{e}_1$ and then integrate the whole system.

Remark 4.2.2 Equations 4.2.5 tell us that the components of $\mathbf{e}_1$ solve the Sturm-Liouville equation

$$z''(u, v) + l(v) z(u, v) = 0 \tag{4.2.6}$$

for $z(u, v)$. The study of these solutions, called Sturm-Liouville theory, is important for the study of linear partial differential equations [5]. This won't affect our calculations here, but it is interesting.

Unfortunately, it's not possible to solve this system analytically in full generality. So, we consider some special cases.

4.2.1 Case 1: $f(v) = l(v) = 0$

In the case that $f(v) = l(v) = 0$, our integrable system reduces to

$$\begin{aligned} d\mathbf{X} &= \mathbf{e}_1 du + \mathbf{e}_2 dv \\ d\mathbf{e}_1 &= \mathbf{e}_3 dv \\ d\mathbf{e}_2 &= \mathbf{e}_3 du \\ d\mathbf{e}_3 &= 0 \end{aligned}$$

We can integrate $d\mathbf{e}_3$ to find that

$$\begin{aligned}
\mathbf{e}_3 &= \mathbf{c}_1 \\
\Rightarrow d\mathbf{e}_2 &= \mathbf{c}_1 du \\
\Rightarrow \mathbf{e}_2 &= \mathbf{c}_1 u + \mathbf{c}_2 \\
\texttt{and}\ d\mathbf{e}_1 &= \mathbf{c}_1 dv \\
\Rightarrow \mathbf{e}_1 &= \mathbf{c}_1 v + \mathbf{c}_3,
\end{aligned}$$

where $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are all constant vectors. We can then solve for $\mathbf{X}$.

$$\begin{aligned}
d\mathbf{X} &= [\mathbf{c}_1 v + \mathbf{c}_3]\, du + [\mathbf{c}_1 u + \mathbf{c}_2]\, dv \\
\Rightarrow \frac{\partial \mathbf{X}}{\partial u} &= \mathbf{c}_1 v + \mathbf{c}_3 \\
\Rightarrow \mathbf{X}(u,v) &= \mathbf{c}_1 uv + \mathbf{c}_3 u + \mathbf{a}(v) \\
\texttt{and}\ \frac{\partial \mathbf{X}}{\partial v} &= \mathbf{c}_1 u + \mathbf{c}_2 \\
\Rightarrow \mathbf{X}(u,v) &= \mathbf{c}_1 uv + \mathbf{c}_2 v + \mathbf{b}(u) \\
\Rightarrow \mathbf{X}(u,v) &= \mathbf{c}_1 uv + \mathbf{c}_2 v + \mathbf{c}_3 u + \mathbf{c}_4, \qquad (4.2.7)
\end{aligned}$$

where $\mathbf{a}(v), \mathbf{b}(u)$ are arbitrary functions of v and u respectively, and $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$, and $\mathbf{c}_4$ are arbitrary vector-valued constants. The choice of constants

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{c}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.2.8)$$

is an acceptable affine frame, and it yields a familiar surface: the saddle.

$$SADDLE(u,v) = [uv, u, v]. \qquad (4.2.9)$$

4.2.2 Automation

We can't solve the system in closed form for general $l(v)$ and $f(v)$. However, we *can* automate the solution process for a *given* $l(v)$ and $f(v)$. An example procedure to solve the system in Maple follows.

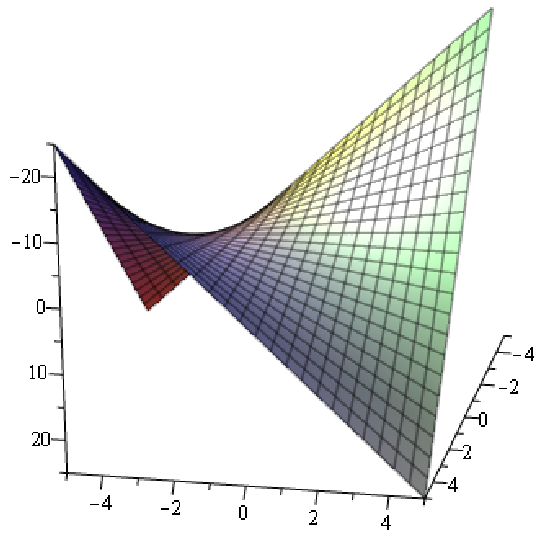

Figure 4.1: The solution to the case $l(v) = f(v) = 0$ is a saddle.

Listing 4.1: Maple Procedure to Solve for Affine Flat and Minimal Surfaces

```
FlatMinFrame:=proc(L,F)
local e,x,sys,soln;
    # This just makes things a little easier to read.
    declare(e[1](u,v),e[2](u,v),e[3](u,v),x(u,v));
    # It's important to force that e1 and e3 are independent of u.
    sys:=[diff(e[3](u,v),v)=L*e[1](u,v),diff(e[3](u,v),u)=0,
            diff(e[1](u,v),u)=0,diff(e[1](u,v),v)=e[3](u,v),
            diff(e[2](u,v),v)=e[1](u,v)*(u*L+F),diff(e[2](u,v),u)=e[3](u,v),
            diff(x(u,v),u)=e[1](u,v),diff(x(u,v),v)=e[2](u,v)];
    soln:=pdsolve(sys);
    simplify(soln);
end proc;
```

The remainder of this section will cover some examples of affine flat, minimal surfaces that were solved for using Listing 1. In the following examples, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$ are arbitrary vector-valued constants. All plots and concrete parametrizations use the constants specified in (4.2.8).

4.2.3 Case 2: $l(v) = k_1$, $f(v) = k_2$, k_1, k_2 constants.

If we allow $l(v)$ and $f(v)$ to be the constant functions k_1 and k_2, we find that

$$\mathbf{X}(u,v) = k_1^{-3/2}\left(k_1 u + k_2\right)\left(e^{\sqrt{k_1}v}\mathbf{c}_1 - e^{-\sqrt{k_1}v}\mathbf{c}_2\right) + v\mathbf{c}_3 + \mathbf{c}_4. \tag{4.2.10}$$

If we let $k_1 = k_2 = k$, and substitute in the constants defined in (4.2.8), we find that

$$\mathbf{X}(u,v) = \left[\frac{e^{\sqrt{k}v}(u+1)}{\sqrt{k}}, -\frac{e^{-\sqrt{k}v}(u+1)}{\sqrt{k}}, v\right]. \tag{4.2.11}$$

If we let $k_1 = k_2 = k = 2$ and plot (4.2.11) with $u \in [-10, 10], v \in [-2, 2]$, we get the plot in figure 4.2.

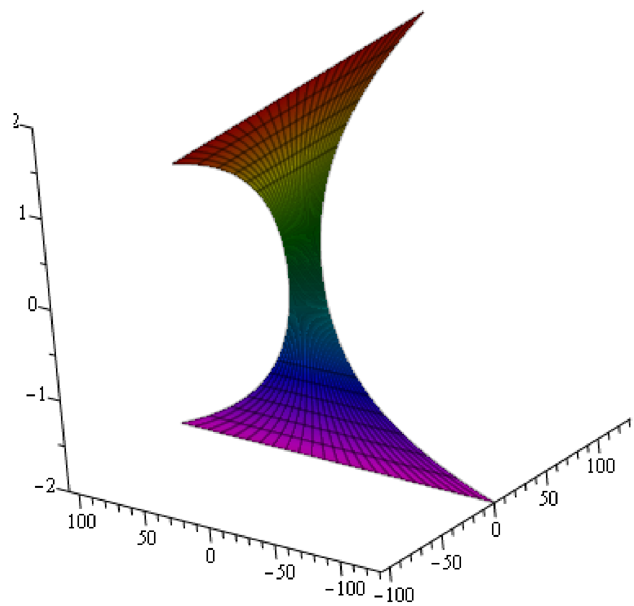

Figure 4.2: The solution to the case $l(v) = f(v) = 2$.

If we enforce that $k_1 = k_2 = -|k|$, then our exponentials change into sines and cosines:

$$\mathbf{X}(u,v) = \frac{(u+1)}{\sqrt{k}} \left[-\mathbf{c}_1 \cos(\sqrt{k}v) + \mathbf{c}_2 \sin(\sqrt{k}v) \right] + v. \tag{4.2.12}$$

And in the $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ basis, we have that

$$\mathbf{X}(u,v) = \left[-\frac{\cos(\sqrt{k}v)(u+1)}{\sqrt{k}}, \frac{\sin(\sqrt{k}v)(u+1)}{\sqrt{k}}, v \right]. \tag{4.2.13}$$

This formula a parametrization of a helicoid. If we let $k_1 = k_2 = -2$ and plot (4.2.13) with $u \in [-10, 10]$, $v \in [0, 2\pi]$, we get the plot in figure 4.3.

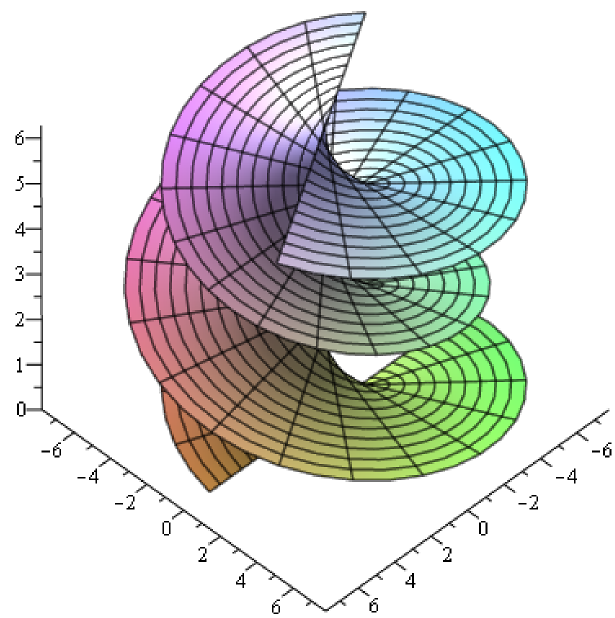

Figure 4.3: The solution to the case $l(v) = f(v) = -2$.

4.2.4 Case 3: $l(v) = v$, $f(v) = 0$.

If we let $l(v) = v$ and $f(v) = 0$, we get some sense of the l dependence of our system. In this case, we find that

$$\mathbf{X}(u,v) = u\sqrt{v} \left[I_{-1/3}(2v^{3/2}/3)\mathbf{c}_1 - K_{1/3}(2v^{3/2}/3)\mathbf{c}_2 \right] + v\mathbf{c}_3 + \mathbf{c}_4, \tag{4.2.14}$$

where I_α and K_α are the modified Bessel functions of the first and second kind respectively. If we substitute in the constants defined in (4.2.8), we find that

$$\mathbf{X}(u,v) = \left[u\sqrt{v}I_{-1/3}(2v^{3/2}/3), -u\sqrt{v}K_{1/3}(2v^{3/2}/3), v\right]. \tag{4.2.15}$$

If we plot this function for $u \in [-10, 10], v \in [-3, 3]$, we get the surface in figure 4.4, which is very reminiscent of figure 4.2, albeit displayed from a different angle.

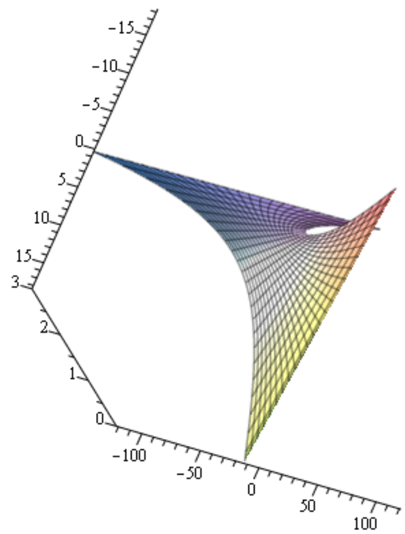

Figure 4.4: The solution to the case $l(v) = v, f(v) = 0$.

4.2.5 Case 4: $l(v) = -v, f(v) = 0$.

If we let $l(v) = -v$ and $f(v) = 0$, we get something qualitatively similar to the $k_1 = k_2 = -|k|$ case. We find that

$$\mathbf{X}(u,v) = -u\sqrt{v}\left[\mathbf{c}_1 J_{-1/3}(2v^{3/2}/3) + \mathbf{c}_2 Y_{-1/3}(2v^{3/2}/3)\right] + \mathbf{c}_3 v, \tag{4.2.16}$$

where J_α and Y_α are the unmodified Bessel functions of the first and second kind respectively.

Remark 4.2.3 Since the modified Bessel functions are the extensions of the Bessel functions into the complex plane, the relationship between case 3 and case 4 makes a lot of sense.

In the $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ basis, we have that

$$\mathbf{X}(u,v) = \left[-u\sqrt{v}J_{-1/3}(2v^{3/2}/3), -u\sqrt{v}Y_{-1/3}(2v^{3/2}/3), v\right]. \tag{4.2.17}$$

If we plot $\mathbf{X}$ for $u \in [-10, 10]$ and $v \in [0, 2\pi]$ we get the surface in 4.5, which looks very similar to figure 4.3.

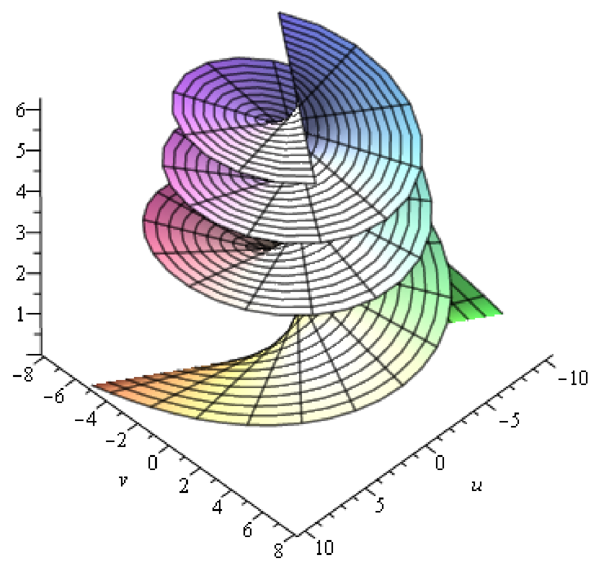

Figure 4.5: The solution to the case $l(v) = -v,\ f(v) = 0$.

4.2.6 Case 5: $l(v) = 0, f(v) = v$.

If we instead let $l(v) = 0$ and $f(v) = v$, we get some sense of the f dependence of our system. In this case, we find that

$$\mathbf{X}(u,v) = \left(uv + \frac{1}{12}v^4\right)\mathbf{c}_1 + \left(u + \frac{1}{6}v^3\right)\mathbf{c}_2 + v\mathbf{c}_3 + \mathbf{c}_4. \tag{4.2.18}$$

In the $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ basis, this is

$$\mathbf{X}(u,v) = \left[uv + \frac{1}{12}v^4, u + \frac{1}{6}v^3 + v\right]. \tag{4.2.19}$$

If we plot this for $u \in [-10, 10], v \in [-5, 5]$, we get the surface in figure 4.6. The case for $l(v) = 0$, $f(v) = -v$ is qualitatively similar.

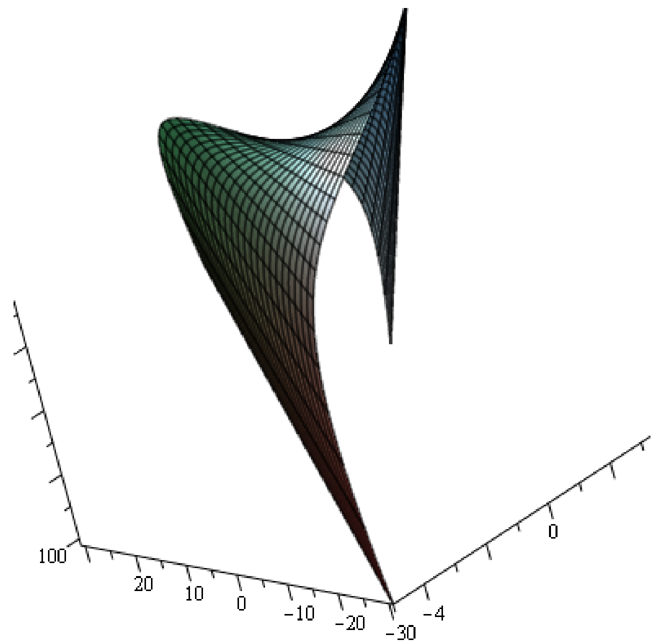

Figure 4.6: The solution to the case $l(v) = 0, f(v) = v$.

4.2.7 Observations

It seems that $l(v)$ produces hyperbolic Bessel-like behavior, while $f(v)$ produces high-order polynomial-like behavior. It is possible to solve the system analytically for much more nontrivial $l(v)$ and $f(v)$, including l and f as arbitrary polynomials. However, the results are messy and not very informative. The plots look about the same, but more difficult functions begin to appear. The hypergeometric function appears in the l dependence and the Gamma function appears in the f dependence.

Chapter 5

Conclusion and Outlook

Our investigation of hyperbolic affine flat affine minimal surfaces yields unexpectedly rich structure. Since we get a different surface for each choice of $l(v)$ and $f(v)$, there exists an infinite-dimensional family of these surfaces parametrized by 2 arbitrary functions of 1 variable. Since there is only one Euclidean flat, Euclidean minimal surface—the plane—this is a surprising result!

Hyperbolic affine flat, affine minimal surfaces are also a special case of surfaces that can be subjected to affine Bäcklund transformations [1]. The Sine-Gorden equation is the following nonlinear wave equation:

$$-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial t^2} = \sin(\psi(x,t)). \tag{5.0.1}$$

Suppose that $\psi(x,t)$ is a solution to equation 5.0.1. Let $a \in \mathbb{R}$. Then the system

$$\begin{aligned} \phi_x &= \psi_x + 2a\sin\left(\frac{\psi+\phi}{2}\right) && (5.0.2) \\ \phi_t &= -\psi_t + \frac{2}{a}\sin\left(\frac{\phi-\psi}{2}\right) && (5.0.3) \end{aligned}$$

is solvable for $\phi(x,t)$, which will also be a solution of equation 5.0.1. Other, more complicated transformations are also possible.

The Swedish mathematician Albert Victor Bäcklund discovered a bijective correspondance between solutions to the Sine-Gordon equation and Euclidean surfaces of constant negative Gauss curvature [1]. Since solutions to Sine-Gordon can be transformed into each other, this gives us an algorithm for generating new Euclidean surfaces that meet the same conditions.

In 1980, Shiing-Shen Chern and Chuu-Lian Terng investigated the analog of this correspondence for affine surfaces. They discovered that a similar correspondence exists so long as the surfaces are hyperbolic and affine minimal [1]. Our hyperbolic affine flat affine minimal surfaces are thus a special case of the class of surfaces investigated by Chern and Terng.

Although Chern and Terng proved the correspondence for hyperbolic affine minimal surfaces, explicit parametrizations for these surfaces are difficult to come by and very few examples of the affine Bäcklund transformation have been generated since then. By characterizing a special family of these surfaces, we have generated a large class of potential examples of affine Bäcklund transforms. The explicit PDE system we found for these surfaces offers the tantalizing possibility that it might be possible to find an explicit formula for these Bäcklund transforms in terms of the functions $f(v)$ and $l(v)$.

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