## A Tale of Two Arc Lengths

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- Euclidean arc length and the Frenet equations
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- Definition of Euclidean and equiaffine spaces
- Euclidean arc length and the Frenet equations
- Affine arc length
- The affine first fundamental form
- Two notions of arc length!
- Main theorem

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- Euclidean arc length and the Frenet equations
- Affine arc length
- The affine first fundamental form
- Two notions of arc length!
- Main theorem
- Examples

Euclidean space $\mathbb{E}^{3}$ is the vector space $\mathbb{R}^{3}$ together with an inner product $\langle$,$\rangle which is defined on all tangent vectors \mathbf{v}, \mathbf{w}$ to all points $\mathbf{x} \in \mathbb{R}^{3}$.

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The symmetry group $E(3)$ of $\mathbb{E}^{3}$ consists of all maps $\phi: \mathbb{E}^{3} \rightarrow \mathbb{E}^{3}$ of the form

$$
\phi(\mathbf{x})=A \mathbf{x}+\mathbf{b}
$$

where $A \in S O(3)$ (or $O(3)$ ) and $\mathbf{b} \in \mathbb{R}^{3}$.

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These are precisely the maps that preserve the inner product: if $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}} \mathbb{E}^{3}$, then

$$
\left\langle\phi_{*}(\mathbf{v}), \phi_{*}(\mathbf{w})\right\rangle=\langle A \mathbf{v}, A \mathbf{w}\rangle=\langle\mathbf{v}, \mathbf{w}\rangle .
$$

All metric notions for submanifolds of $\mathbb{E}^{3}$ are defined in terms of this inner product:

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- If $\alpha: I \rightarrow \mathbb{E}^{3}$ is a regular curve, then the arc length function along $\alpha$ is

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$$

- If $\Sigma \subset \mathbb{E}^{3}$ is a regular surface, then the first fundamental form of $\Sigma$ is defined for any tangent vector $\mathbf{v}$ to $\Sigma$ as

$$
\mathrm{I}_{\mathrm{Euc}}(\mathbf{v})=\langle\mathbf{v}, \mathbf{v}\rangle
$$

Equiaffine space (or, more simply, affine space) $\mathbb{A}^{3}$ is the vector space $\mathbb{R}^{3}$ together with a volume form $d V: \Lambda^{3}\left(\mathbb{R}^{3}\right) \rightarrow \mathbb{R}$.

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$$
d V\left(\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right)=\operatorname{det}\left(\left[\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right]\right) .
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$$

The symmetry group $A(3)$ of $\mathbb{A}^{3}$ consists of all maps $\phi: \mathbb{A}^{3} \rightarrow \mathbb{A}^{3}$ of the form

$$
\phi(\mathbf{x})=A \mathbf{x}+\mathbf{b},
$$

where $A \in S L(3)$ and $\mathbf{b} \in \mathbb{R}^{3}$.

This means that the volume of parallelpipeds is preserved, but the overall length of and angle between vectors doesn't have to be.


Given two points on a surface, we can apply an affine transformation to change the Euclidean distance between the two points.


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However, certain aspects of these notions can be generalized to the affine setting.

## Affine arc length for curves in $\mathbb{A}^{3}$

If $\alpha$ is a nondegenerate curve in $\mathbb{E}^{3}$ (i.e., if the vectors $\alpha^{\prime}(t), \alpha^{\prime \prime}(t)$ are linearly independent for all $\left.t\right)$, then we can associate to each point of $\alpha$ the Frenet frame $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ defined by:

$$
\mathbf{e}_{1}(t)=\frac{\alpha^{\prime}(t)}{\left\|\alpha^{\prime}(t)\right\|}, \quad \mathbf{e}_{2}(t)=\frac{\mathbf{e}_{1}^{\prime}(t)}{\left\|\mathbf{e}_{1}^{\prime}(t)\right\|}, \quad \mathbf{e}_{3}(t)=\mathbf{e}_{1}(t) \times \mathbf{e}_{2}(t),
$$

where for any vector $\mathbf{v},\|\mathbf{v}\|=\sqrt{\langle\mathbf{v}, \mathbf{v}\rangle}$.

The Frenet frame is a set of orthonormal basis vectors which completely describe a curve up to rigid motion. It allows us dispense with a generalized coordinate system.


Key properties of the Frenet frame:

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- It is invariant under the action of $E(3):$ if $\phi \in E(3)$ and $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ is the Frenet frame associated to a curve $\alpha$, then $\left(\phi_{*}\left(\mathbf{e}_{1}\right), \phi_{*}\left(\mathbf{e}_{2}\right), \phi_{*}\left(\mathbf{e}_{3}\right)\right)$ is the Frenet frame associated to $\phi(\alpha)$.


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- It is constructed from the derivatives of $\alpha$ : the vectors $\mathbf{e}_{1}(t), \mathbf{e}_{2}(t)$ are obtained by applying the Gram-Schmidt orthogonalization process to $\alpha^{\prime}(t), \alpha^{\prime \prime}(t)$, and then $\mathbf{e}_{3}(t)$ is the unique vector that makes $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ an oriented orthonormal basis for $\mathbb{E}^{3}$.


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- The matrix $A$ whose columns are the Frenet vectors $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ is an element of $S O(3)$.

The arc length function $\bar{s}(t)$ provides an "ideal" parametrization for a nondegenerate curve $\alpha$ in $\mathbb{E}^{3}$ : if we let $t(\bar{s})$ denote the inverse function for $\bar{s}(t)$ and reparametrize $\alpha$ as

$$
\alpha(\bar{s})=\alpha(t(\bar{s})),
$$

then the Frenet frame of $\alpha$ is simply:

$$
\mathbf{e}_{1}(\bar{s})=\alpha^{\prime}(\bar{s}), \quad \mathbf{e}_{2}(\bar{s})=\frac{\alpha^{\prime \prime}(\bar{s})}{\left\|\alpha^{\prime \prime}(\bar{s})\right\|}, \quad \mathbf{e}_{3}(\bar{s})=\mathbf{e}_{1}(\bar{s}) \times \mathbf{e}_{2}(\bar{s}) .
$$

The $\kappa(\bar{s})$ and torsion $\tau(\bar{s})$ of $\alpha$ are defined by the Frenet equations:

$$
\begin{aligned}
& \mathbf{e}_{1}^{\prime}(\bar{s})=\kappa(\bar{s}) \mathbf{e}_{2}(\bar{s}) \\
& \mathbf{e}_{2}^{\prime}(\bar{s})=-\kappa(\bar{s}) \mathbf{e}_{1}(\bar{s})+\tau(\bar{s}) \mathbf{e}_{3}(\bar{s}) \\
& \mathbf{e}_{3}^{\prime}(\bar{s})=-\tau(\bar{s}) \mathbf{e}_{2}(\bar{s}) .
\end{aligned}
$$

Now suppose that $\alpha(t)$ is a curve in $\mathbb{A}^{3}$. What would be the right analog for the Frenet frame for $\alpha$ ?

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Ideally it should be a trio of vectors $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ with the properties that:

- It should be invariant under the action of $A(3)$.
- It should be constructed from the derivatives of $\alpha$.
- The matrix $A$ whose columns are the frame vectors $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ should be an element of $S L(3)$.

Since there are no notions of length or inner product for tangent vectors in $\mathbb{A}^{3}$, we might start by simply choosing

$$
\overline{\mathbf{e}}_{1}(t)=\alpha^{\prime}(t), \quad \overline{\mathbf{e}}_{2}(t)=\alpha^{\prime \prime}(t), \quad \overline{\mathbf{e}}_{3}(t)=\alpha^{\prime \prime \prime}(t)
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$$

These vectors satisfy the first two conditions above, but not necessarily the third.

The easiest way to remedy this issue is to scale all three vectors simultaneously. This suggests the following definitions:

- A regular curve $\alpha: I \rightarrow \mathbb{A}^{3}$ is called nondegenerate if the vectors $\alpha^{\prime}(t), \alpha^{\prime \prime}(t), \alpha^{\prime \prime \prime}(t)$ are linearly independent for all $t \in I$.
- A regular curve $\alpha: I \rightarrow \mathbb{A}^{3}$ is called nondegenerate if the vectors $\alpha^{\prime}(t), \alpha^{\prime \prime}(t), \alpha^{\prime \prime \prime}(t)$ are linearly independent for all $t \in I$.
- The affine Frenet frame for $\alpha$ is the trio of vectors $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}\right)$ defined by

$$
\begin{aligned}
& \mathbf{e}_{1}(t)=\frac{\alpha^{\prime}(t)}{\sqrt[3]{\operatorname{det}\left[\alpha^{\prime}(t) \alpha^{\prime \prime}(t) \alpha^{\prime \prime \prime}(t)\right]}}, \\
& \mathbf{e}_{2}(t)=\frac{\alpha^{\prime \prime}(t)}{\sqrt[3]{\operatorname{det}\left[\alpha^{\prime}(t) \alpha^{\prime \prime}(t) \alpha^{\prime \prime \prime}(t)\right]}}, \\
& \mathbf{e}_{3}(t)=\frac{\alpha^{\prime \prime \prime}(t)}{\sqrt[3]{\operatorname{det}\left[\alpha^{\prime}(t) \alpha^{\prime \prime}(t) \alpha^{\prime \prime \prime}(t)\right]}}
\end{aligned}
$$

By analogy with the Euclidean case, an optimal parametrization for $\alpha$ would be given by a function $s_{\alpha}(t)$ with the property that

$$
\operatorname{det}\left[\alpha^{\prime}\left(s_{\alpha}\right) \quad \alpha^{\prime \prime}\left(s_{\alpha}\right) \quad \alpha^{\prime \prime \prime}\left(s_{\alpha}\right)\right] \equiv 1
$$

so that we could simply define

$$
\mathbf{e}_{1}\left(s_{\alpha}\right)=\alpha^{\prime}\left(s_{\alpha}\right), \quad \mathbf{e}_{2}\left(s_{\alpha}\right)=\alpha^{\prime \prime}\left(s_{\alpha}\right), \quad \mathbf{e}_{3}\left(s_{\alpha}\right)=\alpha^{\prime \prime \prime}\left(s_{\alpha}\right)
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$$

This function $s_{\alpha}(t)$ is called the affine arc length function along $\alpha$, and it is given by

$$
s_{\alpha}(t)=\int_{0}^{t} \sqrt[6]{\operatorname{det}\left[\alpha^{\prime}(\sigma) \alpha^{\prime \prime}(\sigma) \alpha^{\prime \prime \prime}(\sigma)\right]} d \sigma
$$

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- Unlike Euclidean arc length, which depends only on the first derivative of $\alpha$, the affine arc length depends on the first three derivatives of $\alpha$. In general, this number is dependent on the dimension of the ambient affine space: the affine arc length of a curve $\alpha: I \rightarrow \mathbb{A}^{n}$ depends on the first $n$ derivatives of $\alpha$.

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- Unlike Euclidean arc length, which depends only on the first derivative of $\alpha$, the affine arc length depends on the first three derivatives of $\alpha$. In general, this number is dependent on the dimension of the ambient affine space: the affine arc length of a curve $\alpha: I \rightarrow \mathbb{A}^{n}$ depends on the first $n$ derivatives of $\alpha$.
- The affine arc length is only nonzero for nondegenerate curves; so for instance, any curve contained in a plane in $\mathbb{A}^{3}$ has affine arc length zero according to this definition. It may, however, have nonzero affine arc length when regarded as a curve in $\mathbb{A}^{2}$.

What does the affine arc length really mean?

## What does the affine arc length really mean?

Proposition 1: Let $\alpha: I \rightarrow \mathbb{R}^{3}$ be a nondegenerate curve (in the affine sense). Let $\bar{s}(t), \kappa(t), \tau(t)$ be the Euclidean arc length, curvature, and torsion of $\alpha$, respectively, and suppose that $\tau(t)>0$. Then the affine arc length $s_{\alpha}(t)$ of $\alpha$ is given by

$$
s_{\alpha}(t)=\int_{0}^{t} \sqrt[6]{\kappa(\sigma)^{2} \tau(\sigma)} \bar{s}^{\prime}(\sigma) d \sigma
$$

In particular, if $\alpha$ is parametrized by its Euclidean arc length $\bar{s}$, then

$$
s_{\alpha}(\bar{s})=\int_{0}^{\bar{s}} \sqrt[6]{\kappa(\sigma)^{2} \tau(\sigma)} d \sigma
$$

Although the Euclidean quantities $\kappa(t), \tau(t), \bar{s}(t)$ are not individually preserved by the action of the equiaffine group, this proposition yields the following corollary:

Although the Euclidean quantities $\kappa(t), \tau(t), \bar{s}(t)$ are not individually preserved by the action of the equiaffine group, this proposition yields the following corollary:

Corollary: The Euclidean 1-form $d s_{\alpha}=\sqrt[6]{\kappa^{2} \tau} d \bar{s}$ associated to a nondegenerate curve $\alpha: I \rightarrow \mathbb{R}^{3}$ is invariant under the action of the equiaffine group.

Affine first fundamental form for surfaces in $\mathbb{A}^{3}$
If $\Sigma \subset \mathbb{E}^{3}$ is a regular surface with parametrization $X: \mathbb{R}^{2} \rightarrow \mathbb{E}^{3}$, then the first fundamental form of $\Sigma$ is given by

$$
\mathrm{I}_{\mathrm{Euc}}=E d u^{2}+2 F d u d v+G d v^{2}
$$

where $E=\left\langle X_{u}, X_{u}\right\rangle, F=\left\langle X_{u}, X_{v}\right\rangle, G=\left\langle X_{v}, X_{v}\right\rangle$.

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where $E=\left\langle X_{u}, X_{u}\right\rangle, \quad F=\left\langle X_{u}, X_{v}\right\rangle, G=\left\langle X_{v}, X_{v}\right\rangle$.
$\mathrm{I}_{\text {Euc }}$ is a quadratic form on each tangent plane $T_{\mathbf{x}} \Sigma$ that expresses the Euclidean metric on $T_{\mathbf{x}} \Sigma$ with respect to the basis $\left(X_{u}, X_{v}\right)$.

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- It is invariant under the action of $E(3)$.
- It is constructed from derivatives of $X$.
- It is invariant under reparametrizations of $\Sigma$ : if $\bar{X}(\bar{u}, \bar{v})=X(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v}))$, then

$$
\begin{aligned}
\mathrm{I}_{\mathrm{Euc}} & =\bar{E} d \bar{u}^{2}+2 \bar{F} d \bar{u} d \bar{v}+\bar{G} d \bar{v}^{2} \\
& =E d u^{2}+2 F d u d v+G d v^{2}
\end{aligned}
$$

We also associate to $\Sigma$ the second fundamental form, which measures the curvature of $\Sigma$.


We write the second fundamental form as

$$
\mathrm{II}_{\mathrm{Euc}}=e d u^{2}+2 f d u d v+g d v^{2}
$$

where

$$
e=\left\langle X_{u u}, N\right\rangle, \quad f=\left\langle X_{u v}, N\right\rangle, \quad g=\left\langle X_{v v}, N\right\rangle
$$

and $N$ is a unit normal vector field to $\Sigma$.

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$$

and $N$ is a unit normal vector field to $\Sigma$.
The second fundamental form encapsulates the curvature properties of the surface; in particular, the Gauss curvature of the surface is

$$
K=\frac{\operatorname{det}\left(\mathrm{I}_{\mathrm{Euc}}\right)}{\operatorname{det}\left(\mathrm{I}_{\mathrm{Euc}}\right)}=\frac{e g-f^{2}}{E G-F^{2}}
$$

$\mathrm{II}_{\text {Euc }}$ is not necessarily positive definite, but otherwise it shares the same key properties as $\mathrm{I}_{\text {Euc }}$.

Now let $\Sigma \subset \mathbb{A}^{3}$ be a regular surface with parametrization $X: U \rightarrow \mathbb{A}^{3}$. Can we find an analog for the Euclidean first fundamental form which is invariant under the action of the equiaffine group?

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- It should be invariant under the action of $A(3)$.
- It should be constructed from the derivatives of $X$.
- It should be invariant under reparametrizations of $\Sigma$.

The quadratic form satisfying these conditions is called the affine first fundamental form $\mathrm{I}_{\text {aff }}$, and it is constructed as follows: set

$$
\begin{aligned}
\ell & =\operatorname{det}\left[\begin{array}{lll}
X_{u} & X_{v} & X_{u u}
\end{array}\right], \\
m & =\operatorname{det}\left[\begin{array}{lll}
X_{u} & X_{v} & X_{u v}
\end{array}\right], \\
n & =\operatorname{det}\left[\begin{array}{lll}
X_{u} & X_{v} & X_{v v}
\end{array}\right],
\end{aligned}
$$

and define

$$
\mathrm{I}_{\mathrm{aff}}=\left|\ell n-m^{2}\right|^{-1 / 4}\left(\ell d u^{2}+2 m d u d v+n d v^{2}\right)
$$

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and define

$$
\mathrm{I}_{\mathrm{aff}}=\left|\ell n-m^{2}\right|^{-1 / 4}\left(\ell d u^{2}+2 m d u d v+n d v^{2}\right)
$$

Since $\mathrm{I}_{\mathrm{aff}}$ is a quadratic form, it can be used to define a metric on the surface $\Sigma$. Unlike in the Euclidean case, this metric is not necessarily positive definite; it may be positive or negative definite, or indefinite.

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- A regular surface $\Sigma$ with parametrization $X: \mathbb{R}^{2} \rightarrow \mathbb{A}^{3}$ is called nondegenerate if the quadratic form $\ell d u^{2}+2 m d u d v+n d v^{2}$ is nondegenerate (i.e., if $\ell n-m^{2} \neq 0$ ).

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- A nondegenerate parametrized surface is called elliptic if the quadratic form $\mathrm{I}_{\mathrm{aff}}$ is definite and hyperbolic if $\mathrm{I}_{\mathrm{aff}}$ is indefinite.

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Proposition 2: Let $X: \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$ be a regular parametrization for a nondegenerate surface $\Sigma$. Let $\mathrm{II}_{\text {Euc }}$ denote the Euclidean second fundamental form of $\Sigma$, and let $K$ denote the Euclidean Gauss curvature of $\Sigma$. Then

$$
\mathrm{I}_{\mathrm{aff}}=|K|^{-1 / 4} \mathrm{I} \mathrm{I}_{\mathrm{Euc}} .
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$$
\mathrm{I}_{\mathrm{aff}}=|K|^{-1 / 4} \mathrm{II}_{\mathrm{Euc}} .
$$

Corollary: The Euclidean quadratic form $|K|^{-1 / 4} \mathrm{II}_{\text {Euc }}$ associated to a nondegenerate surface $\Sigma \subset \mathbb{R}^{3}$ is invariant under the action of the equiaffine group.

Curves in surfaces: two notions of arc length!
Now suppose that $\alpha: I \rightarrow \mathbb{A}^{3}$ is a curve whose image is contained in a nondegenerate surface $\Sigma=X\left(\mathbb{R}^{2}\right) \subset \mathbb{A}^{3}$.

Curves in surfaces: two notions of arc length!
Now suppose that $\alpha: I \rightarrow \mathbb{A}^{3}$ is a curve whose image is contained in a nondegenerate surface $\Sigma=X\left(\mathbb{R}^{2}\right) \subset \mathbb{A}^{3}$.

The restriction of $\mathrm{I}_{\text {aff }}$ to $\alpha$ defines an arc length function $s_{\Sigma}$ along $\alpha$, as follows:

$$
s_{\Sigma}(t)=\int_{0}^{t} \sqrt{\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(\sigma)\right)} d \sigma
$$

We will refer to the function $s_{\Sigma}$ on $\alpha$ as the "induced arc length" function from $\Sigma$.

## Curves in surfaces: two notions of arc length!

Now suppose that $\alpha: I \rightarrow \mathbb{A}^{3}$ is a curve whose image is contained in a nondegenerate surface $\Sigma=X\left(\mathbb{R}^{2}\right) \subset \mathbb{A}^{3}$.

The restriction of $\mathrm{I}_{\text {aff }}$ to $\alpha$ defines an arc length function $s_{\Sigma}$ along $\alpha$, as follows:

$$
s_{\Sigma}(t)=\int_{0}^{t} \sqrt{\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(\sigma)\right)} d \sigma
$$

We will refer to the function $s_{\Sigma}$ on $\alpha$ as the "induced arc length" function from $\Sigma$.

Although the affine arc length $s_{\alpha}$ and the induced arc length $s_{\Sigma}$ are both "metric functions" along $\alpha$, they may or may not agree, even for fairly trivial examples.

Example 1: Let $\Sigma \subset \mathbb{R}^{3}$ be the unit sphere, with the parametrization

$$
X(u, v)=[\cos (u) \cos (v), \sin (u) \cos (v), \sin (v)] .
$$

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Regarded as a surface in $\mathbb{E}^{3}, \Sigma$ has uniform Gauss curvature $K=1$ and second fundamental form

$$
\mathrm{II}_{\mathrm{Euc}}=\cos ^{2}(v) d u^{2}+d v^{2}=\mathrm{I}_{\mathrm{Euc}} .
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$$

Therefore the affine first fundamental form of $\Sigma$ is

$$
\mathrm{I}_{\mathrm{aff}}=\left|K^{-1 / 4}\right| \mathrm{II}_{\mathrm{Euc}}=\cos ^{2}(v) d u^{2}+d v^{2}=\mathrm{I}_{\mathrm{Euc}} .
$$

Example 1a: Let $\alpha$ be the great circle

$$
\alpha(t)=X(t, 0)=[\cos (t), \sin (t), 0] .
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Because $\Sigma$ has the property that $\mathrm{II}_{\text {Euc }}=\mathrm{I}_{\text {Euc }}$, the induced arc length function $s_{\Sigma}(t)$ agrees with the Euclidean arc length function $\bar{s}(t)$; therefore,

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s_{\Sigma}(t)=t
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$$

But because $\alpha$ is contained in a plane, it is degenerate as a curve in $\mathbb{A}^{3}$ and its affine arc length function $s_{\alpha}(t)$ is identically equal to zero.

Example 1b: Let $\alpha$ be the "spherical helix" curve

$$
\alpha(t)=X(8 t, t)=[\cos (8 t) \cos (t), \sin (8 t) \cos (t), \sin (t)] .
$$

Example 1b: Let $\alpha$ be the "spherical helix" curve

$$
\alpha(t)=X(8 t, t)=[\cos (8 t) \cos (t), \sin (8 t) \cos (t), \sin (t)] .
$$



As in the previous example, the induced arc length function $s_{\Sigma}(t)$ is equal to the Euclidean arc length function

$$
s_{\Sigma}(t)=\int_{0}^{t} \sqrt{1+64 \cos ^{2}(\sigma)} d \sigma
$$

while the affine arc length function $s_{\alpha}(t)$ is

$$
s_{\alpha}(t)=\int_{0}^{t} \sqrt[6]{48 \cos (\sigma)\left(43+672 \cos ^{2}(\sigma)\right)} d \sigma
$$

The two arc length functions are qualitatively similar, although their integrands are noticeably different:



Question: Which nondegenerate curves $\alpha$ in a nondegenerate surface $\Sigma$ have the property that the two arc length functions $s_{\alpha}(t), s_{\Sigma}(t)$ are equal? We will call a curve with this property commensurate.

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Theorem: Let $X: U \rightarrow \mathbb{R}^{3}$ be a regular parametrization for a nondegenerate surface $\Sigma$, and let $\alpha: I \rightarrow \mathbb{R}^{3}$ be a regular, nondegenerate curve contained in $\Sigma$. Then $\alpha$ is a commensurate curve if and only if, for all $t \in I$,

$$
\operatorname{det}\left[\begin{array}{lll}
\alpha^{\prime}(t) & \alpha^{\prime \prime}(t) & \alpha^{\prime \prime \prime}(t) \tag{1}
\end{array}\right]=\left[\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(t)\right)\right]^{3}
$$

Proof: The two arc length functions are defined by

$$
\begin{aligned}
& s_{\alpha}(t)=\int_{0}^{t} \sqrt[6]{\operatorname{det}\left[\alpha^{\prime}(\sigma) \alpha^{\prime \prime}(\sigma) \alpha^{\prime \prime \prime}(\sigma)\right]} d \sigma \\
& s_{\Sigma}(t)=\int_{0}^{t} \sqrt{\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(\sigma)\right)} d \sigma
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$$

These two functions are equal if and only if the integrands are equal; i.e., if and only if

$$
\begin{aligned}
& \sqrt[6]{\operatorname{det}\left[\alpha^{\prime}(t) \alpha^{\prime \prime}(t) \alpha^{\prime \prime \prime}(t)\right]}=\sqrt{\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(t)\right)} \\
& \Leftrightarrow \quad \operatorname{det}\left[\begin{array}{lll}
\alpha^{\prime}(t) & \alpha^{\prime \prime}(t) & \alpha^{\prime \prime \prime}(t)
\end{array}\right]=\left[\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(t)\right)\right]^{3} .
\end{aligned}
$$

Corollary 1: Let $X: U \rightarrow \mathbb{R}^{3}$ be a regular parametrization for a nondegenerate surface $\Sigma$, and let $\alpha: I \rightarrow \mathbb{R}^{3}$ be a regular, nondegenerate curve contained in $\Sigma$. Then $\alpha$ is a commensurate curve if and only if, for all $t \in I$,

$$
\kappa(t)^{2} \tau(t)=\left(|K(t)|^{-1 / 4} k_{n}(t)\right)^{3}
$$

where

- $\kappa(t), \tau(t)$ are the Euclidean curvature and torsion functions of $\alpha$;
- $K(t)$ is the Gauss curvature of $\Sigma$ at the point $\alpha(t)$;
- $k_{n}(t)$ is the normal curvature of $\Sigma$ at the point $\alpha(t)$ in the direction of $\alpha^{\prime}(t)$.

How plentiful are commensurate curves?

## How plentiful are commensurate curves?

Corollary 2: Let $X: U \rightarrow \mathbb{R}^{3}$ be a regular parametrization for a nondegenerate surface $\Sigma$. Given any point $\mathbf{x} \in \Sigma$ and any tangent vector $\mathbf{v} \in T_{\mathbf{x}} \Sigma$ for which $\mathrm{I}_{\mathrm{aff}}(\mathbf{v}) \neq 0$, there exists a 1-parameter family of commensurate curves $\alpha$ in $\Sigma$ such that $\alpha(0)=\mathbf{x}$ and $\alpha^{\prime}(0)=\mathbf{v}$.

Proof: Let $\mathbf{x}=X\left(u_{0}, v_{0}\right)$ and $\mathbf{v}=a X_{u}+b X_{v}$. We can write

$$
\alpha(t)=X(u(t), v(t))
$$

for some smooth functions $u(t), v(t)$.

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for some smooth functions $u(t), v(t)$.
The condition (1) is invariant under reparametrizations of $\alpha$, so without loss of generality we may assume (locally) that $u(t)=u_{0}+a t$, and therefore

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Equation (1) then becomes a 3rd-order nonlinear ODE for the function $v(t)$.

For $t=0$, we have

$$
\alpha(0)=X\left(u_{0}, v(0)\right), \quad \alpha^{\prime}(0)=a X_{u}+v^{\prime}(0) X_{v}
$$

So the conditions $\alpha(0)=\mathbf{x}, \alpha^{\prime}(0)=\mathbf{v}$ are equivalent to the initial conditions

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v(0)=v_{0}, \quad v^{\prime}(0)=b
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$$

for the function $v(t)$.
The local existence/uniqueness theorem for ODEs guarantees that for any real number $c$, there exists a unique local solution to (1) with

$$
v(0)=v_{0}, \quad v^{\prime}(0)=b, \quad v^{\prime \prime}(0)=c
$$

For example, if we set $u_{0}=0, a=1$, so that

$$
\alpha(t)=X(t, v(t)),
$$

then equation (1) becomes:

$$
\begin{aligned}
\operatorname{det}\left[\frac{d}{d t} X(t, v(t)) \frac{d^{2}}{d t^{2}} X(t, v(t))\right. & \left.\frac{d^{3}}{d t^{3}} X(t, v(t))\right] \\
& =\left[\mathrm{I}_{\mathrm{aff}}\left(\frac{d}{d t} X(t, v(t))\right)\right]^{3} .
\end{aligned}
$$

Example 1c: Let $\Sigma \subset \mathbb{R}^{3}$ be the unit sphere, and let $\alpha$ be a commensurate curve on $\Sigma$. For simplicity, assume that $\alpha$ is parametrized by its Euclidean arc length $\bar{s}$.

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Since all normal curvatures on $\Sigma$ are equal to 1 , Corollary 1 implies that the curve $\alpha(\bar{s})$ on $\Sigma$ is commensurate if and only if its curvature and torsion satisfy

$$
\kappa(\bar{s})^{2} \tau(\bar{s}) \equiv 1
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Since all normal curvatures on $\Sigma$ are equal to 1 , Corollary 1 implies that the curve $\alpha(\bar{s})$ on $\Sigma$ is commensurate if and only if its curvature and torsion satisfy

$$
\kappa(\bar{s})^{2} \tau(\bar{s}) \equiv 1
$$

Moreover, the fact that $\alpha$ lies on the unit sphere implies that

$$
\left(\frac{1}{\kappa(\bar{s})}\right)^{2}+\left(\frac{1}{\tau(\bar{s})} \frac{d}{d s}\left(\frac{1}{\kappa(\bar{s})}\right)\right)^{2}=1
$$

Together, these two equations imply that $\kappa(\bar{s})$ satisfies the ODE

$$
\left(\kappa^{\prime}(\bar{s})\right)^{2}=\frac{\kappa(\bar{s})^{2}-1}{\kappa(\bar{s})^{2}} .
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$$

Since $\kappa(\bar{s})$ is assumed to be positive and $\bar{s}$ is only well-defined up to an additive constant, we may assume without loss of generality that

$$
\kappa(\bar{s})=\sqrt{\bar{s}^{2}+1},
$$

and then we have

$$
\tau(\bar{s})=\frac{1}{\bar{s}^{2}+1}
$$

Unfortunately the corresponding Frenet equations cannot be integrated analytically, but we can integrate them numerically:

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Every other commensurate curve on the sphere can be obtained by rotating and translating this one.

This is a rather interesting curve! The geodesic curvature $\kappa_{g}(\bar{s})$ function of a curve in a regular surface $\Sigma \subset \mathbb{R}^{3}$ is defined by the condition that

$$
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$$
\kappa^{2}=\kappa_{g}^{2}+k_{n}^{2}
$$

Since all normal curvatures on the sphere are equal to 1 and $\kappa(\bar{s})=\sqrt{\bar{s}^{2}+1}$, this curve has geodesic curvature function

$$
\kappa_{g}(\bar{s})=s
$$

Thus we may regard the commensurate curves on the sphere as spherical analogs of plane curves with curvature function

$$
\kappa(\bar{s})=s
$$



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- first as the solution to an elasticity problem posed in 1694 by James Bernoulli;
- again in work of Augustin Fresnel in 1816 regarding the problem of light diffracting through a slit;
- yet again in work of Arthur Talbot in 1901 related to designing railroad tracks so as to provide as smooth a riding experience as possible.


## More examples:

We compute examples of commensurate curves on various surfaces as follows: we assume that the curve is parametrized as

$$
\alpha(t)=X(u(t), v(t)),
$$

where

$$
\begin{equation*}
u^{\prime}(t)=\cos (\theta(t)), \quad v^{\prime}(t)=\sin (\theta(t)) \tag{2}
\end{equation*}
$$

for some unknown function $\theta(t)$.

Then, for a given parametrization $X(u, v)$ of $\Sigma$, the equation

$$
\operatorname{det}\left[\begin{array}{lll}
\alpha^{\prime}(t) & \alpha^{\prime \prime}(t) & \alpha^{\prime \prime \prime}(t)
\end{array}\right]=\left[\mathrm{I}_{\mathrm{aff}}\left(\alpha^{\prime}(t)\right)\right]^{3}
$$

for commensurate curves becomes a second-order ODE for the function $\theta(t)$, with coefficients depending on the functions $u(t), v(t)$.

We numerically solve the system consisting of this ODE together with equations (2) for various choices of initial conditions in order to generate the curves in the following examples.

Example 2: The paraboloid

$$
X(u, v)=\left[v \cos (u), v \sin (u), v^{2}\right]
$$



Example 3: The hyperbolic paraboloid

$$
X(u, v)=[u, v, u v]
$$

Example 4: The hyperboloid

$$
X(u, v)=[\cos (u)-v \sin (u), \sin (u)+v \cos (u), v]
$$



Example 5: The helicoid

$$
X(u, v)=[u \cos (v), u \sin (v), v]
$$



